École Polytechnique
Fédérale de Lausanne

Analysis I

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Contents

Chapter 1

Prerequisites and basic notions

1.1 Basics

We assume that the reader is familiar with the following notions.

1.1.1 Algebraic identities

- $(a+b)^2 = a^2 + 2ab + b^2$
- $a^2 - b^2 = (a+b)(a-b)$
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

1.1.2 Exponents

Let $a, b \in \mathbb{R}$ and $n, m \in \mathbb{N}$.

- $a^n \cdot a^m = a^{n+m}$
- $a^n \cdot b^n = (ab)^n$
- $\frac{a^n}{a^m} = a^{n-m}$
- $a^0 = 1$ if $a \neq 0$
- $(a^n)^m = a^{n \cdot m}$
- $\sqrt[n]{a} = a^{1/n}$
- $(\frac{a}{b})^n = \frac{a^n}{b^n}$
- $a^1 = a$

1.1.3 Logarithms

Let $a, b > 0$, and $c \in \mathbb{R}$.

- $\log(ab) = \log(a) + \log(b)$
- $\log(\frac{a}{b}) = \log(a) - \log(b)$

- $\log(a^c) = c \cdot \log(a)$
- $\log(1) = 0$

1.1.4 Trigonometry

The functions $\sin(x), \cos(x), \tan(x)$ are defined for $x \in \mathbb{R}$. We have the following identities:

- $\tan(x) = \frac{\sin(x)}{\cos(x)}$ if $\cos(x) \neq 0$
- $\cot(x) = \frac{\cos(x)}{\sin(x)}$ if $\sin(x) \neq 0$
- $\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$
- $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$

1.2 Functions

The elementary functions include polynomials, rational functions, algebraic functions, exponential functions and trigonometric functions.

Definition 1.2.1 (Polynomial functions).
A polynomial is a function of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $a_i \in \mathbb{R}$ and $n \in \mathbb{N}$.

Example. Examples of polynomial functions include:

- $P(x) = 2x^3 - 4x + 7$ (cubic polynomial)
- $P(x) = x^2 - 5x + 6$ (quadratic polynomial)
- $P(x) = 3x + 1$ (linear polynomial)
- $P(x) = 5$ (constant polynomial)

Definition 1.2.2 (Rational functions).
A rational function is a function of the form $R(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

Example. Examples of rational functions include:

- $R(x) = \frac{x^2-1}{x+2}$
- $R(x) = \frac{2x^3+3x}{x^2-4}$
- $R(x) = \frac{1}{x-5}$

Definition 1.2.3 (Algebraic functions).

An algebraic function is a function that can be expressed as the root of a polynomial equation with coefficients that are polynomials.

Example. Examples of algebraic functions include:

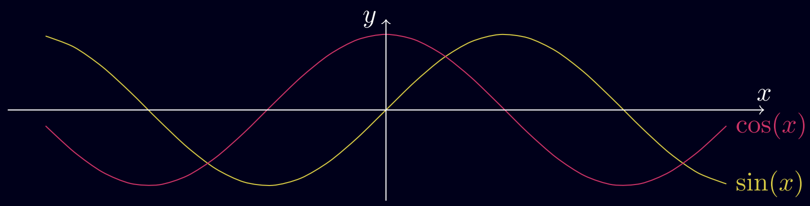
- $f(x) = \sqrt{x^2 + 1}$
- $f(x) = \sqrt[3]{x^3 - 2x + 1}$
- $f(x) = \frac{\sqrt{x}}{x+1}$

Definition 1.2.4 (Transcendental functions).
A transcendental function is a function that is not algebraic, such as exponential and trigonometric functions.

Example. Examples of transcendental functions include:

- $f(x) = e^x$ (exponential function)
- $f(x) = \sin(x)$ (trigonometric function)
- $f(x) = \ln(x)$ (logarithmic function)

Below is the graph of the $\sin(x)$ and $\cos(x)$ trigonometric functions.



1.2.1 Graphs transformations

Definition 1.2.5 (Graph transformations).
Graph transformations involve shifting, stretching, compressing, and reflecting the graphs of functions. The general form of a transformed function can be expressed as:

$$g(x) = a \cdot f(b \cdot (x - c)) + d$$

where:

- a affects the vertical stretch/compression and reflection.
- b affects the horizontal stretch/compression and reflection.
- c affects the horizontal shift.
- d affects the vertical shift.

Example. Consider the function $f(x) = \sin(x)$. The following transformations illustrate the effects of the parameters:

- $g(x) = 2 \cdot \sin(x)$ (vertical stretch by a factor of 2)
- $g(x) = \sin(2x)$ (horizontal compression by a factor of 2)
- $g(x) = \sin(x - \frac{\pi}{2})$ (horizontal shift to the right by $\frac{\pi}{2}$)
- $g(x) = \sin(x) + 1$ (vertical shift upwards by 1)

1.2.2 Type of functions

Let A and B be two sets.

Definition 1.2.6 (Domain and image of a function).
The domain of a function $f : A \to B$ is the set A where for each element of $a \in A$, the function is defined.
The image (or codomain) of the function is the set B. For every $a \in A$, there exists a unique $b \in B$ such that $f(a) = b$.

Definition 1.2.7 (Surjective functions).
A function $f : A \to B$ is surjective (onto) if for every $b \in B$, there exists at least one $a \in A$ such that $f(a) = b$.

Definition 1.2.8 (Injective functions).
A function $f : A \to B$ is injective (one-to-one) if for every $b \in B$, there exists at most one $a \in A$ such that $f(a) = b$.

Definition 1.2.9 (Bijective functions).
A function $f : A \to B$ is bijective if it is both injective and surjective. This means that for every $b \in B$, there exists exactly one $a \in A$ such that $f(a) = b$.

1.2.3 Inverse functions

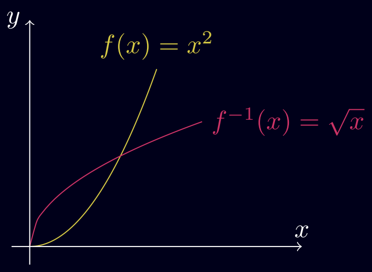
Let $f : A \to B$ be a bijective function.

Definition 1.2.10 (Inverse function).
The inverse function of f, denoted by $f^{-1} : B \to A$, is defined such that for every $b \in B$, $f^{-1}(b) = a$ where $f(a) = b$.

Example. Consider the function $f(x) = x^2$ defined on $\mathbb{R}$. This function is bijective on $[0, +\infty)$.
To find the inverse function, we solve for x in terms of y:

$$y = x^2 \implies x = \sqrt{y}$$

Thus, the inverse function is $f^{-1}(y) = \sqrt{y}$. We can plot both functions to visualize their relationship.



Chapter 2

Real Numbers

2.1 The Field of Real Numbers

Chapter 3

Complex numbers

The equation $x^2+1=0$ has no solution in $\mathbb{R}$. To solve this problem, we introduce a new number i such that $i^2=-1$.

Proof. Let's prove that the equation $x^2+1=0$ has no solution in $\mathbb{R}$. Assume that there exists $x \in \mathbb{R}$ such that $x^2+1=0$. Then, we have:

$$x^2 = -1$$

However, the left-hand side is non-negative for all $x \in \mathbb{R}$, while the right-hand side is negative. This is a contradiction. Therefore, there are no solutions in $\mathbb{R}$. $\square$

3.1 Definition and properties of complex numbers

Definition 3.1.1 (Complex numbers).
The set of complex numbers is defined as $\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$ where i is the imaginary unit with the property that $i^2=-1$. Addition and mutliplication of complex numbers are defined as follows:

- $(a+bi)+(c+di) = (a+c)+(b+d)i$
- $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$

Let also add that:

- $\exists 0 \in \mathbb{C} : 0 = 0+0i$ such as $(a+bi)+(0+0i) = (a+bi) \quad \forall a, b \in \mathbb{R}$.
- $\exists$ an inverse for $(a+bi) : ((-a)+(-b)i)+(a+bi) = 0+0i = 0 \quad \forall a, b \in \mathbb{R}$.
- $\exists 1 \in \mathbb{C} : 1 = 1+0i$ such as $(a+bi)(1+0i) = (a+bi) \quad \forall a, b \in \mathbb{R}$.
- $\forall z \in \mathbb{C} : z \neq 0 \implies \exists z^{-1} \in \mathbb{C} : zz^{-1} = 1$.

Definition 3.1.2 (Inverse of a complex number).

Let $z = a + bi \in \mathbb{C}$ such as $z \neq 0$ (i.e., $a^2 + b^2 \neq 0$). The inverse of z is given by:

$$z^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$

Let's verify that $zz^{-1} = 1$:

$$zz^{-1} = (a + bi)\left(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i\right) = \frac{a^2 + b^2}{a^2 + b^2} = 1$$

Example. Let's verify that complex numbers respect the distributive property.
Let $z_1 = (a_1 + b_1 i)$, $z_2 = (a_2 + b_2 i)$ and $z_3 = (a_3 + b_3 i)$ be three complex numbers. We want to show that:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

Let's compute left side:

$$z_1(z_2 + z_3) = (a_1 + b_1 i)((a_2 + a_3) + (b_2 + b_3)i)$$

$$= (a_1(a_2 + a_3) - b_1(b_2 + b_3)) + (a_1(b_2 + b_3) + b_1(a_2 + a_3))i$$

Now, let's compute the right side:

$$z_1 z_2 + z_1 z_3 = (a_1 a_2 - b_1 b_2 + a_1 a_3 - b_1 b_3) + (a_1 b_2 + b_1 a_2 + a_1 b_3 + b_1 a_3)i$$

Both sides are equal, thus $\mathbb{C}$ is a commutative field.

We can prove that $\mathbb{C}$ is not an ordered field. Indeed, if we assume that $i > 0$, then $i^2 = -1 > 0$ which is a contradiction. If we assume that $i < 0$, then $-i > 0$ and $(-i)^2 = -1 > 0$ which is also a contradiction. Thus, $\mathbb{C}$ is not an ordered field.

3.2 Form of complex numbers

Complex numbers can be represented in 3 different forms.

3.2.1 Algebraic form

Definition 3.2.1 (Algebraic form).
The algebraic form of a complex number $z = a + bi$ where $a, b \in \mathbb{R}$. Which can also be written as:

$$z = \mathrm{Re}(z) + \mathrm{Im}(z)i$$

where $\mathrm{Re}(z) = a$ is the real part of z and $\mathrm{Im}(z) = b$ is the imaginary part of z.

The modulus of a complex number $z = a + bi$ is defined as:

$$|z| = \sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2} = \sqrt{a^2 + b^2} \geq 0$$

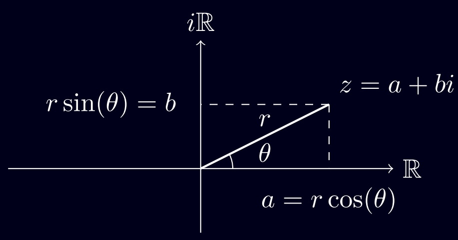
$$|z| = 0 \iff z = 0$$

3.2.2 Trigonometric form

Definition 3.2.2 (Trigonometric form).
The trigonometric form of a complex number $z = a + bi$ is given by:

$$z = r(\cos(\theta) + i\sin(\theta)) \quad r \geq 0, \theta \in \mathbb{R}$$

where:



and:

$$\mathrm{Re}(z) = a = r\cos(\theta), \quad \mathrm{Im}(z) = b = r\sin(\theta)$$

The modulus of z is given by:

$$|z| = r = \sqrt{r^2\cos^2(\theta) + r^2\sin^2(\theta)} = r \geq 0$$

The argument of z ($z \neq 0$) is given by:

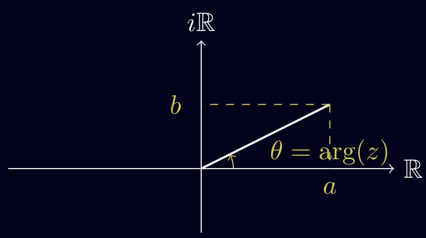
$$r \neq 0 \implies \sin(\theta) = \frac{\mathrm{Im}(z)}{r}, \quad \cos(\theta) = \frac{\mathrm{Re}(z)}{r}$$

Showing that θ is defined up to an additive factor of $2k\pi$, $k \in \mathbb{Z}$. Also:

$$\tan(\theta) = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} = \frac{b}{a} \quad \text{if } a \neq 0$$

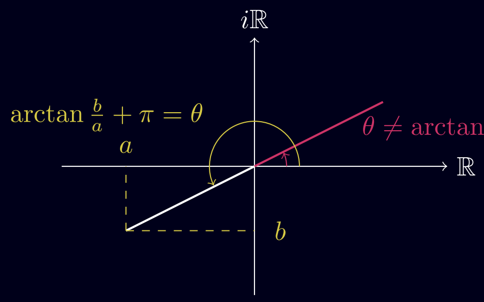
Example. Let's find the argument of $z = a + bi$. We have:
If $a > 0$, then:

$$\theta = \arctan\left(\frac{b}{a}\right) + 2k\pi, \quad k \in \mathbb{Z}$$



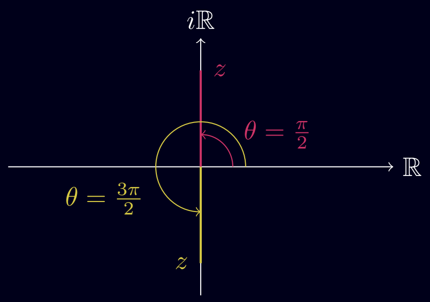
If $a < 0$, then:

$$\theta = \arctan\left(\frac{b}{a}\right) + (2k+1)\pi, \quad k \in \mathbb{Z}$$



Finally, if $a = 0$, then:

$$\theta = \begin{cases} \frac{\pi}{2} + 2k\pi, & b > 0 \\ \frac{3\pi}{2} + 2k\pi, & b < 0 \end{cases}$$



The argument of z is only defined for $z \neq 0$.

3.2.3 Exponential form

Definition 3.2.3 (Exponential form)**.**
The exponential form of a complex number $z = a + bi$ is given by:

$$e^z = \exp(z) = e^a(\cos(b) + i\sin(b))$$

If $z = a$, then:

$$e^z = e^a(\cos(0) + i\sin(0)) = e^a(1 + 0) = e^a$$

If $z = bi$, then:

$$e^z = e^0(\cos(b) + i\sin(b)) = e^{ib}$$

Definition 3.2.4 (Euler's formula)**.**
For any real number $\theta \in \mathbb{R}$, we have:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Let's show that $e^{i(\theta_1+\theta_2)} = e^{i\theta_1} \cdot e^{i\theta_2}$:

$$e^{i(\theta_1+\theta_2)} = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$$

$$= (\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2))$$
$$= (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) = e^{i\theta_1} \cdot e^{i\theta_2}$$

Let's also show that an imaginary exponential is periodic with period 2π:

$$y_1 = y_2 + 2k\pi,\ k \in \mathbb{Z} \implies e^{iy_1} = e^{iy_2+2k\pi} = e^{iy_2}(\cos(2k\pi) + i\sin(2k\pi)) = e^{iy_2}$$

If we take a complex number in trigonometric form $z = r(\cos(\theta) + i\sin(\theta))$, we can write it in exponential form as:

$$z = re^{i\theta}$$

3.2.4 Summary of the forms of a complex number

A complex number can be represented as:

$$\begin{aligned} z &= \mathrm{Re}(z) + i \cdot \mathrm{Im}(z) && \text{(Algebraic form)} \\ &= |z|(\cos(\arg(z)) + i\sin(\arg(z))) && \text{(Trigonometric form)} \\ &= |z|e^{i\arg(z)} && \text{(Exponential form)} \end{aligned}$$

Usually complex numbers are represented in polar form (trigonometric or exponential form).

Example. Let's show that the modulus of $z = e^{(e^{i\phi})}$ is equal to $e^{\cos(\phi)}$.

$$z = e^{(e^{i\phi})} = e^{(\cos(\phi)+i\sin(\phi))} = e^{\cos(\phi)}e^{i\sin(\phi)} = e^{\cos(\phi)} \cdot (\cos(\sin(\phi)) + i\sin(\sin(\phi)))$$

Thus, the modulus of z is:

$$|z| = \sqrt{(e^{\cos(\phi)})^2 \cdot [(\cos(\sin(\phi)))^2 + (\sin(\sin(\phi)))^2]} = e^{\cos(\phi)}$$

Example. Let's find the exponential form of $z = -1 + i$. We need to calculate the modulus and the argument of z:

$$|z| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

Since $\mathrm{Re}(z) = -1 < 0$ and $\mathrm{Im}(z) = 1 > 0$, we are in the second case of the argument calculation:

$$\theta = \arctan\left(\frac{1}{-1}\right) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$$

Thus, the exponential form of z is:

$$z = \sqrt{2}e^{i\frac{3\pi}{4}}$$

Example. Let's find the algebraic form of $z = 5e^{i\frac{2\pi}{3}}$. We have:

$$z = 5\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = 5\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -\frac{5}{2} + i\frac{5\sqrt{3}}{2}$$

3.2.5 Operations in polar form

Polar form is very useful to perform operations on complex numbers.

Definition 3.2.5 (Mutiplication in polar form).
To multiply two complex numbers in polar form, we multiply their moduli and add their arguments. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Proof. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. We have:

$$z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}$$

By using the property of exponentials, we get:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

□

Example. Let $z = e^{i\phi}$, $w = e^{i\psi}$, $|z| = 1$ and $|w| = 1$. Then:

$$zw = e^{i(\phi+\psi)}$$

Definition 3.2.6 (Division in polar form).
To divide two complex numbers in polar form, we divide their moduli and subtract their arguments. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \quad \text{if } z_2 \neq 0$$

Definition 3.2.7 (Inverse in polar form).
The inverse of a complex number, $z \neq 0$ in polar form is given by:

$$z^{-1} = \frac{1}{r} e^{-i\theta} \quad \text{if } z = re^{i\theta} \text{ and } r > 0$$

Proof. Let $z = re^{i\theta}$ such that $r > 0$. We have:

$$zz^{-1} = re^{i\theta} \cdot \frac{1}{r} e^{-i\theta} = 1 \cdot e^{i(\theta-\theta)} = 1 \cdot e^0 = 1$$

□

Definition 3.2.8 (De Moivre's theorem).
For any complex number $z = re^{i\theta}$ and any integer $n \in \mathbb{N}^*$, we have:

$$z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta))$$

Proof. Let's prove this by induction. For $n = 1$, we have:

$$z^1 = r^1 e^{i\cdot 1\cdot\theta} = re^{i\theta}$$

Now, let's assume it's true for $n = k$, i.e., $z^k = r^k e^{ik\theta}$. For $n = k+1$, we have:

$$z^{k+1} = z^k z = (r^k e^{ik\theta})(re^{i\theta}) = r^k r e^{ik\theta} e^{i\theta} = r^{k+1} e^{i(k+1)\theta}$$

Thus, by induction, the theorem holds for all $n \in \mathbb{N}^*$. □

3.2.6 The Most Beautiful Formula in Mathematics

Let $y = \pi$, then:

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + 0 = -1$$

Theorem 3.2.1 (Euler's identity).

$$e^{i\pi} + 1 = 0$$

This formula links the 5 most important constants in mathematics: $e, i, \pi, 1, 0$.

3.3 Conjugate of a complex number

Definition 3.3.1 (Conjugate of a complex number).
The conjugate of a complex number $z = a + bi$ is defined as:

$$\overline{z} = a - bi$$

or in polar form for $z = re^{i\theta}$:

$$\overline{z} = re^{-i\theta} = r(\cos(-\theta) + i\sin(-\theta)) = r(\cos(\theta) - i\sin(\theta))$$

If $\bar{z} \neq 0$, then:

$$z\overline{z} = (a+bi)(a-bi) = a^2 + b^2 = |z|^2$$

3.3.1 Properties of the conjugate

Let $z_1, z_2 \in \mathbb{C}$ and $\lambda \in \mathbb{R}$, we have:

- $\overline{\overline{z}} = z$
- $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ if $z_2 \neq 0$
- $\overline{\lambda z} = \lambda \overline{z}$

Also, we have:

- $\text{Re}(z) = \frac{z+\overline{z}}{2}$
- $\text{Im}(z) = \frac{z-\overline{z}}{2i}$

If $|z| = 1$, then in polar form for $z = \cos(\phi) + i\sin(\phi)$, we have:

$$\cos(\phi) = \frac{e^{i\phi} + e^{-i\phi}}{2}, \quad \sin(\phi) = \frac{e^{i\phi} - e^{-i\phi}}{2i}$$

Example. Let's express $(\sin(\phi))^4$:

$$(\sin(\phi))^4 = \left(\frac{e^{i\phi} - e^{-i\phi}}{2i}\right)^4 = \frac{(e^{i\phi} - e^{-i\phi})^4}{16i^4} = \frac{e^{4i\phi} - 4e^{2i\phi} + 6 - 4e^{-2i\phi} + e^{-4i\phi}}{16}$$

Since $-4e^{2i\phi} - 4e^{-2i\phi} = -8\cos(2\phi)$ and $e^{4i\phi} + e^{-4i\phi} = 2\cos(4\phi)$, we get:

$$(\sin(\phi))^4 = \frac{3}{8} - \frac{1}{2}\cos(2\phi) + \frac{1}{8}\cos(4\phi)$$

3.4 Roots of complex numbers

Definition 3.4.1 (n-th roots of a complex number).
Let $z \in \mathbb{C}^*$ and $n \in \mathbb{N}^*$. The n-th roots of z are the solutions of the equation:

$$w^n = z$$

If we write z in polar form as $z = re^{i\theta}$, then the n-th roots of z are given by:

$$w_k = r^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}} \qquad k = 0, 1, \ldots, n-1$$

There are n distinct n-th roots of z.

Proof. Let $w = \rho e^{i\phi}$ be an n-th root of z. We have:

$$w^n = (\rho e^{i\phi})^n = \rho^n e^{in\phi} = re^{i\theta}$$

By comparing the moduli and arguments, we get:

$$\rho^n = r \implies \rho = r^{\frac{1}{n}}$$

and

$$n\phi = \theta + 2k\pi \implies \phi = \frac{\theta + 2k\pi}{n} \qquad k \in \mathbb{Z}$$

Since the argument is defined up to an additive factor of 2π, we only need to consider $k = 0, 1, \ldots, n-1$ to get all distinct n-th roots. Thus, the n-th roots of z are given by:

$$w_k = r^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}} \qquad k = 0, 1, \ldots, n-1$$

□

Example. Let $z^2 = w = re^{i\theta}$ (where $r > 0$) be a complex number. Then:

$$z_k = r^{\frac{1}{2}} e^{i\frac{\theta+2k\pi}{2}} \qquad k = 0, 1$$

Thus, the two square roots of w are:

$$z_0 = r^{\frac{1}{2}} e^{i\frac{\theta}{2}}, \quad z_1 = r^{\frac{1}{2}} e^{i\left(\frac{\theta}{2}+\pi\right)} = -r^{\frac{1}{2}} e^{i\frac{\theta}{2}} = -z_0$$

More explicitly:

$$(z_1)^2 = (z_2)^2 = w$$

Example. Let's solve the equation $z^3 = -1 + i$ ($z \in \mathbb{C}$). Let $w = -1 + i = \sqrt{2}e^{i\frac{3\pi}{4}}$ be the complex number. We have:

$$z_k = (\sqrt{2})^{\frac{1}{3}} e^{i\frac{\frac{3\pi}{4}+2k\pi}{3}} = 2^{\frac{1}{6}} e^{i\left(\frac{\pi}{4}+\frac{2k\pi}{3}\right)} \quad k = 0, 1, 2$$

Thus, the three cube roots of w are:

$$z_0 = 2^{\frac{1}{6}} e^{i\frac{\pi}{4}}, \quad z_1 = 2^{\frac{1}{6}} e^{i\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)} = 2^{\frac{1}{6}} e^{i\frac{11\pi}{12}}, \quad z_2 = 2^{\frac{1}{6}} e^{i\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)} = 2^{\frac{1}{6}} e^{i\frac{19\pi}{12}}$$

Example. Let $z \in \mathbb{C}, z \neq 0$. We want to find all complex numbers such that $(\frac{z}{\bar{z}})^3 = \bar{z}$. We have:

$$z = re^{i\theta} \implies \bar{z} = re^{-i\theta}$$

Thus:

$$\left(\frac{z}{\bar{z}}\right)^3 = \left(\frac{re^{i\theta}}{re^{-i\theta}}\right)^3 = (e^{2i\theta})^3 = e^{6i\theta}$$

We want to solve:

$$e^{6i\theta} = re^{-i\theta}$$

By comparing the moduli and arguments, we get:

$$r = 1$$

and

$$6\theta = -\theta + 2k\pi \implies 7\theta = 2k\pi \implies \theta = \frac{2k\pi}{7}, \quad k \in \mathbb{Z}$$

Thus, the solutions are:

$$z_k = e^{i\frac{2k\pi}{7}}, \quad k = 0, 1, \ldots, 6$$

3.5 Polynomials with complex coefficients

Definition 3.5.1 (Polynomial with complex coefficients).
A polynomial with complex coefficients is a function $P : \mathbb{C} \to \mathbb{C}$ defined by:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

where $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$. The degree of the polynomial is n.

In the case of quadratic polynomials, like in the case with real coefficients, we have:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

except that now $a, b, c \in \mathbb{C}$, thus:

$$\sqrt{b^2 - 4ac} \in \mathbb{C} \implies w \in \mathbb{C} : w^2 = b^2 - 4ac$$

Theorem 3.5.1 (Fundamental Theorem of Algebra)**.** Every non-constant polynomial with complex coefficients, $P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ where $a_n \neq 0$ can be factored as:

$$P(z) = a_n(z - z_1)^{m_1}(z - z_2)^{m_2} \ldots (z - z_n)^{m_n}$$

where $z_1, z_2, \ldots, z_n \in \mathbb{C}$ are the distinct roots of $P(z)$ and $m_1, m_2, \ldots, m_n \in \mathbb{N}^*$ $(m_1 + m_2 + \ldots + m_n = n)$ are their respective multiplicities.

Example. Let's check if $P(z) = z^2 + (1-i)z - i$ is divisible by $z - i$. We have:

$$P(i) = i^2 + (1-i)i - i = -1 + i + 1 - i = 0$$

Thus, $P(z)$ is divisible by $z - i$.

3.5.1 Polynomials with real coefficients

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial with real coefficients, then its complex roots come in conjugate pairs. That is, if $z = a + bi$ is a root of $P(x)$, then its conjugate $\bar{z} = a - bi$ is also a root of $P(x)$.

Proof. Let $z = a + bi$ be a root of $P(x)$, then:

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0$$

Taking the complex conjugate of both sides, we get:

$$\overline{P(z)} = \overline{a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0} = 0$$

Since the coefficients are real, this simplifies to:

$$P(\bar{z}) = a_n \bar{z}^n + a_{n-1}\bar{z}^{n-1} + \ldots + a_1 \bar{z} + a_0 = 0$$

Thus, $\bar{z}$ is also a root of $P(x)$. □

Every polynomial with real coefficients of degree $n \geq 2$ can be factored as a product of linear and irreducible quadratic factors with real coefficients.

3.6 Subset of complex numbers

Example. Let $z_o \in \mathbb{C}, r \in \mathbb{R} > 0$. The set of complex numbers defined by:

$$|z - z_0| = r$$

where $z = x + yi$ and $z_0 = x_0 + y_0 i$, can be expressed as:

$$|(x + yi) - (x_0 + y_0 i)| = r$$

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = r$$

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

This is the equation of a circle with center (x_0, y_0) and radius r in the Cartesian plane. Thus, the set of complex numbers z such that $|z - z_0| = r$ represents a circle in the complex plane with center z_0 and radius r.

Example. Let $z \in \mathbb{C}^* : (\frac{|z|}{z})^4 = \frac{1}{\sqrt{2}}(1+i)$. We have:

$$z = re^{i\theta} \implies |z| = r, \quad \frac{|z|}{z} = \frac{r}{re^{i\theta}} = e^{-i\theta}$$

Thus:

$$(e^{-i\theta})^4 = e^{-4i\theta} = \frac{1}{\sqrt{2}}(1+i) = e^{i\frac{\pi}{4}}$$

By comparing the arguments, we get:

$$-4\theta = \frac{\pi}{4} + 2k\pi \implies \theta = -\frac{\pi}{16} - \frac{k\pi}{2}, \quad k = 0, 1, 2, 3$$

Thus, the solutions are:

$$\theta \in \left\{ -\frac{\pi}{16} - \frac{k\pi}{2} \mid k = 0, 1, 2, 3 \right\}$$

and r can take any positive real value. Therefore, the geometric representation of the solutions is given by 4 open half-lines originating from the origin, making angles of $-\frac{\pi}{16}, -\frac{9\pi}{16}, -\frac{17\pi}{16}, -\frac{25\pi}{16}$ with the positive real axis.

3.7 Exercices

This section gathers a selection of exercises related to Chapter 3, taken from weekly assignments, past exams, textbooks, and other sources. The origin of each exercise will be indicated at its beginning.

Chapter 4

Sequences of real numbers

Definition 4.0.1 (Sequences).
A sequence is a function whose domain is the set of natural numbers. In other words, a sequence is an ordered list of numbers, typically denoted as $(a_n)_{n\in\mathbb{N}}$, where each a_n is a real number.

We often denote a sequence by (a_n), $(a_n)_{n\geq 0} = \{a_0, a_1, a_2, \ldots\}$ or simply a_n, where n represents the position of the term in the sequence.

Example. Consider the sequences defined as:

- $a_n = n^2$ (the sequence of perfect squares)
- $b_n = \frac{1}{n}$ (the sequence of reciprocals)
- $c_n = (-1)^n$ (the alternating sequence)
- $f_0 = 0, f_1 = 1, f_{n+2} = f_{n+1} + f_n$ (the Fibonacci sequence)
- $a_n = a \cdot n + b$ (an arithmetic sequence with common difference a and initial term b)
- $a_n = a \cdot r^n$ (a geometric sequence with common ratio r and initial term a)

4.1 Properties of sequences

Definition 4.1.1 (Bounded sequences).
A sequence (a_n) is said to be bounded above if there exists a real number M such that $a_n \leq M$ for all n.
It is bounded below if there exists a real number m such that $a_n \geq m$ for all n.
If a sequence is both bounded above and bounded below, it is called a bounded sequence.

Definition 4.1.2 (Absolute value).

The absolute value of a real number x, denoted by $|x|$, is defined as:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Let be (a_n) a sequence. We say that (a_n) is bounded if there exists $M > 0$ such that $|a_n| \leq M$ for all n.

Definition 4.1.3 (Decreasing and increasing sequences).
A sequence (a_n) is called decreasing if $a_n \geq a_{n+1}$ for all n. It is called increasing if $a_n \leq a_{n+1}$ for all n.

Definition 4.1.4 (Monotonic sequences).
A sequence (a_n) is called monotonic if it is either non-decreasing or non-increasing.

Example. Let's consider the following sequences:

- $a_n = n$ is increasing ($a_{n+1} - a_n = 1 > 0$) and unbounded ($\mathbb{N}$ is not bounded above).
- $a_n = \frac{1}{n+1}$ is decreasing ($a_{n+1} - a_n = -\frac{1}{(n+1)(n+2)} < 0$) and bounded (by 1 and 0).
- $a_n = (-1)^n$ is neither increasing nor decreasing, but it is bounded (by 1 and -1).
- $a_n = a \cdot n + b$ is increasing if $a > 0$, decreasing if $a < 0$, and constant if $a = 0$ ($a_{n+1} - a_n = a(n+1) + b - a \cdot n - b = a$). It is unbounded unless $a = 0$ (if $a > 0$, $\forall S > 0, \forall b \in \mathbb{R}, \exists n \in \mathbb{N} : a_n > S - b$ $(Archimedean) \iff an + b > S)$.
- $f_0 = 0, f_1 = 1, f_{n+2} = f_{n+1} + f_n$ is increasing ($f_{n+2} - f_{n+1} = f_n \geq 0$) and unbounded (we can prove it by induction).

4.2 Recurrence relations

Definition 4.2.1 (Recurrence relation).
A recurrence relation is an equation that defines each term of a sequence in terms of one or more previous terms. It typically consists of:

- An initial condition or base case that specifies the value(s) of the first term(s) of the sequence.
- A formula that relates each term to one or more preceding terms.

A recurrence relation can be visualized by dominoes. Each domino represents a term in the sequence. The initial condition is when we push the first domino and the recurrence relation is how each domino knocks over the next one making it fall.

Example. The Fibonacci sequence is defined by the recurrence relation:

- Initial conditions: $(f_1)^2 - f_2 \cdot f_0 = 1^2 - 1 \cdot 0 = 1$
- Recurrence relation: $(f_{n+1})^2 - f_{n+2} \cdot f_n = f_{n+1}(f_{n-1} + f_n) - f_n(f_n + f_{n+1}) = f_{n+1} \cdot f_{n-1} = f_{n+1} \cdot f_{n-1} - (f_n)^2 = -((f_n)^2)$

Example. Let's find the sum of the first n odd natural numbers.

- $S_1 = 1$
- $S_2 = 1 + 3 = 4$
- $S_3 = 1 + 3 + 5 = 9$
- $S_4 = 1 + 3 + 5 + 7 = 16$

Let hypothesize that $S_n = \sum_{k=1}^{n}(2k-1) = n^2$. We can prove this by induction:

- Base case: $n = 1$, $S_1 = 1^2 = 1$ (true)
- Inductive step: Assume $S_n = n^2$ for some $n \geq 1$. We need to show that $S_{n+1} = (n+1)^2$.

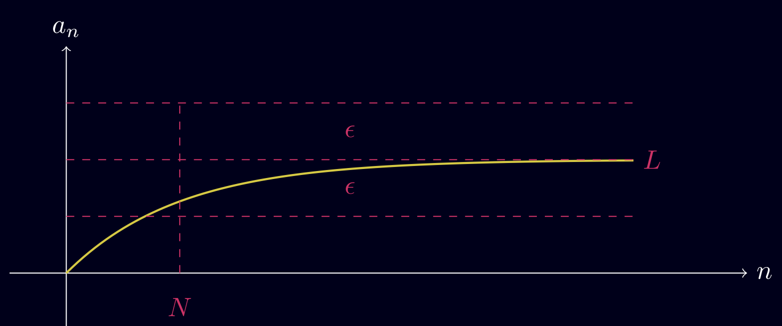
$$S_{n+1} = S_n + (2(n+1) - 1) = n^2 + (2n+1) = n^2 + 2n + 1 = (n+1)^2$$

Thus, by the principle of mathematical induction, the formula holds for all natural numbers n.

4.3 Limits of sequences

Definition 4.3.1 (Limit of a sequence).
A sequence (a_n) is said to converge to a limit L if for every $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$, the absolute difference between a_n and L is less than ϵ. In mathematical notation,

$$\lim_{n\to\infty} a_n = L \iff \forall\epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |a_n - L| < \epsilon.$$



If a sequence does not converge to any limit, it is said to diverge.

Example. Let $a_n = \frac{1}{\sqrt{n+1}}$. We will show that $\lim_{n\to\infty} a_n = 0$.
Let $\epsilon > 0$. We need to find $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - 0| < \epsilon$. We have:

$$|a_n - 0| = \left|\frac{1}{\sqrt{n+1}} - 0\right| = \frac{1}{\sqrt{n+1}} < \epsilon$$

This inequality can be solved as follows:

$$\frac{1}{\sqrt{n+1}} < \epsilon \iff \sqrt{n+1} > \frac{1}{\epsilon} \iff n+1 > \frac{1}{\epsilon^2} \iff n > \frac{1}{\epsilon^2} - 1$$

Thefore, we need n to be greater than $\frac{1}{\epsilon^2} - 1$. Since n must be a natural number, we can choose:

$$N = \left\lceil \frac{1}{\epsilon^2} - 1 \right\rceil$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Thus, for all $n \geq N$, we have:

$$|a_n - 0| = \frac{1}{\sqrt{n+1}} < \epsilon$$

Hence, by the definition of the limit of a sequence, we conclude that $\lim_{n\to\infty} a_n = 0$.

Example. Let's study the limit of the sequence defined by $(a_n) = (-1)^{8^n - 3^n}$. We first need to determine the behavior of the exponent $8^n - 3^n$ as n increases. We want to determine whether $8^n - 3^n$ is even or odd for large n.

- For $n = 0$: $8^0 - 3^0 = 1 - 1 = 0$ (even)
- For $n = 1$: $8^1 - 3^1 = 8 - 3 = 5$ (odd)
- For $n = 2$: $8^2 - 3^2 = 64 - 9 = 55$ (odd)
- For $n = 3$: $8^3 - 3^3 = 512 - 27 = 485$ (odd)

We notice that for all $n \geq 1$, 8^n is even and 3^n is odd, thus $8^n - 3^n$ is always odd. Therefore, for all $n \geq 1$, $a_n = (-1)^{\text{odd}} = -1$. Thefore, the sequence (a_n) is constant and equal to -1 for all $n \geq 1$. Hence, we can conclude that:

$$\lim_{n\to\infty} a_n = -1$$

Example. Let $p \in \mathbb{Q}, p > 0$ and $a_0 = 1, a_n = \frac{1}{n^p}$. Let's show that $\lim_{n\to\infty} a_n = 0$. Let $\epsilon > 0$. We need to find $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - 0| < \epsilon$. We have:

$$|a_n - 0| = \left| \frac{1}{n^p} - 0 \right| = \frac{1}{n^p} < \epsilon$$

This inequality can be solved as follows:

$$\frac{1}{n^p} < \epsilon \iff n^p > \frac{1}{\epsilon} \iff n > \left(\frac{1}{\epsilon}\right)^{\frac{1}{p}}$$

Thefore, we need n to be greater than $\left(\frac{1}{\epsilon}\right)^{\frac{1}{p}}$. Since n must be a natural number, we can choose:

$$N = \left\lceil \left(\frac{1}{\epsilon}\right)^{\frac{1}{p}} \right\rceil$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Thus, for all $n \geq N$, we have:

$$|a_n - 0| = \frac{1}{n^p} < \epsilon$$

Hence, by the definition of the limit of a sequence, we conclude that $\lim_{n\to\infty} a_n = 0$.

Definition 4.3.2 (Uniqueness of limits).
If a sequence (a_n) converges to a limit L, then this limit is unique. In other words, if $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$, then $L_1 = L_2$.

Definition 4.3.3 (Triangle inequality).
For any real numbers x and y, the triangle inequality states that:

$$|x + y| \leq |x| + |y|$$

Proof. Let $x, y \in \mathbb{R}$. We will prove the triangle inequality by considering two casses based on the sign of $x + y$.

- Case 1: $x + y \geq 0$
$$|x + y| = x + y \leq |x| + |y|$$
- Case 2: $x + y < 0$
$$|x + y| = -(x + y) = -x - y \leq |x| + |y|$$

In both cases, we have shown that $|x + y| \leq |x| + |y|$. Therefore, the triangle inequality holds for all real numbers x and y. □

Example. Let's prove that the sequence $a_n = (-1)^n$ does not converge. Let's suppose, for the sake of contradiction, that the sequence converges to a limit L. According to the definition of convergence, for every $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$, $|a_n - L| < \epsilon$. Let's choose $\epsilon = 0.5$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < 0.5$. Now, consider the terms a_N and a_{N+1}:

- If N is even, then $a_N = 1$ and $a_{N+1} = -1$.
- If N is odd, then $a_N = -1$ and $a_{N+1} = 1$.

In either case, we have:

$$|a_N - a_{N+1}| = |1 - (-1)| = 2$$

However, by the triangle inequality, we also have:

$$|a_N - a_{N+1}| \leq |a_N - L| + |L - a_{N+1}| < 0.5 + 0.5 = 1$$

This leads to a contradiction since $2 \leq 1$ is false. Therefore, our initial assumption that the sequence $(a_n) = (-1)^n$ converges must be incorrect. Hence, the sequence $(a_n) = (-1)^n$ does not converge.

Every sequence that converges is bounded. The converse is not true: a bounded sequence does not necessarily converge (e.g., $a_n = (-1)^n$).

Proof. Let (a_n) be a sequence that converges to a limit L. By the definition of convergence, for every $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$, $|a_n - L| < \epsilon$. Let's choose $\epsilon = 1$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < 1$. This implies that:

$$-1 < a_n - L < 1$$

Adding L to all parts of the inequality, we get:

$$L - 1 < a_n < L + 1$$

This shows that for all $n \geq N$, the terms of the sequence (a_n) are bounded between $L-1$ and $L+1$. Now, consider the finite set of terms $a_0, a_1, a_2, \ldots, a_{N-1}$. Since this is a finite set, it has both a maximum and a minimum value. Let:

$$M_1 = \max\{a_0, a_1, a_2, \ldots, a_{N-1}\}$$

and

$$m_1 = \min\{a_0, a_1, a_2, \ldots, a_{N-1}\}$$

Now we can define:

$$M = \max(M_1, L + 1)$$

and

$$m = \min(m_1, L - 1)$$

Thus, for all $n < N$, we have $m_1 \leq a_n \leq M_1$, and for all $n \geq N$, we have $L - 1 < a_n < L + 1$. Therefore, for all $n \in \mathbb{N}$, we have:

$$m < a_n < M$$

This shows that the sequence (a_n) is bounded above by M and bounded below by m. Hence, the sequence (a_n) is bounded. □

4.3.1 Operations on limits

Theorem 4.3.1. Let (a_n) and (b_n) be two sequences that converge to limits L_1 and L_2 respectively. Then:

- The sequence $(a_n \pm b_n)$ converges to $L_1 \pm L_2$.
- The sequence $(a_n \cdot b_n)$ converges to $L_1 \cdot L_2$.
- If $L_2 \neq 0$, the sequence $\left(\frac{a_n}{b_n}\right)$ converges to $\frac{L_1}{L_2}$.
- The sequence $p \cdot (a_n)$ converges to $p \cdot L_1$ for any constant $p \in \mathbb{R}$.

Note that if $(a_n + b_n)$ converges, then either both (a_n) and (b_n) converge, or both diverge. Also if $\lim_{n\to\infty}(a_n - b_n) = 0$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ or both diverge.
When $(a_n \cdot b_n)$ converges, we have:

- $a_n \to L_1$ and $b_n \to L_2$. (e.g., $a_n = \frac{1}{n+1}, b_n = 1$ and $a_n \cdot b_n = \frac{1}{n+1} \to 0$)
- a_n and b_n diverge. (e.g., $a_n = (-1)^n, b_n = (-1)^n + \frac{1}{n+1}$ and $a_n \cdot b_n = 1 + \frac{(-1)^n}{n+1}$)
- $a_n \to L$ and b_n diverges. (e.g., $a_n = (n+1), b_n = \frac{1}{(n+1)^2}$ and $a_n \cdot b_n = \frac{1}{n+1}$)

But if $b_n \to L \neq 0$ then a_n must converge.

4.3.2 Quotients of Polynomial Sequences

Theorem 4.3.2. Let (a_n) and (b_n) be sequences defined by polynomials $P(n)$ and $Q(n)$ respectively, where:

$$a_n = P(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0$$

$$b_n = Q(n) = b_m n^m + b_{m-1} n^{m-1} + \ldots + b_1 n + b_0$$

with $a_k, b_m \neq 0$. The limit of the quotient of these polynomial sequences as n approaches infinity can be determined based on the degrees of the polynomials k and m:

- If $k < m$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.
- If $k = m$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a_k}{b_m}$.
- If $k > m$, then $\frac{a_n}{b_n}$ diverges.

Proof. Let (a_n) and (b_n) be sequences defined by polynomials $P(n)$ and $Q(n)$ respectively, where:

$$a_n = P(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0$$

$$b_n = Q(n) = b_m n^m + b_{m-1} n^{m-1} + \ldots + b_1 n + b_0$$

with $a_k, b_m \neq 0$. We analyze the limit of the quotient $\frac{a_n}{b_n}$ as n approaches infinity based on the degrees of the polynomials k and m:

$$\frac{a_n}{b_n} = \frac{a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0}{b_m n^m + b_{m-1} n^{m-1} + \ldots + b_1 n + b_0} = \frac{n^k}{n^m} \cdot \frac{a_k + \frac{a_{k-1}}{n} + \ldots + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k}}{b_m + \frac{b_{m-1}}{n} + \ldots + \frac{b_1}{n^{m-1}} + \frac{b_0}{n^m}}$$

We easily see that:

$$\lim_{n\to\infty} \frac{a_k + \frac{a_{k-1}}{n} + \ldots + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k}}{b_m + \frac{b_{m-1}}{n} + \ldots + \frac{b_1}{n^{m-1}} + \frac{b_0}{n^m}} = \frac{a_k}{b_m}$$

Thus we have three cases:

- If $k < m$, then $\frac{n^k}{n^m} = \frac{1}{n^{m-k}} \to 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.
- If $k = m$, then $\frac{n^k}{n^m} = 1$. Therefore, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a_k}{b_m}$.
- If $k > m$, then $\frac{n^k}{n^m} = n^{k-m} \to \infty$ as $n \to \infty$. Therefore, the quotient $\frac{a_n}{b_n}$ diverges.

This completes the proof. □

Example. Let $a_n = \frac{3n^2+5n+1}{2n^2+n+10}$. Then we have:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2+5n+1}{2n^2+n+10} = \lim_{n\to\infty} \frac{n^2}{n^2} \cdot \frac{3 + \frac{5}{n} + \frac{1}{n^2}}{2 + \frac{1}{n} + \frac{10}{n^2}} = \frac{3}{2}$$

4.3.3 Squeeze theorem

Theorem 4.3.3. Let (a_n) and (b_n) be two sequences such that $(a_n) \to a$ and $(b_n) \to b$. If there exists a $m_0 \in \mathbb{N} : \forall n \geq m_0 \implies a_n \geq b_n$, then $a \geq b$.

Proof. Let's prove the theorem by contradiction. Suppose that $a < b$. Then, we can choose $\epsilon = \frac{b-a}{2} > 0$. By the definition of convergence, there exist natural numbers N_1 and N_2 such that for all $n \geq N_1$, $|a_n - a| < \epsilon$ and for all $n \geq N_2$, $|b_n - b| < \epsilon$. Let $N = \max(N_1, N_2, m_0)$. Then, for all $n \geq N$, we have:

$$|a_n - a| < \epsilon \implies a_n < a + \epsilon = a + \frac{b-a}{2} = \frac{a+b}{2}$$

and

$$|b_n - b| < \epsilon \implies b_n > b - \epsilon = b - \frac{b-a}{2} = \frac{a+b}{2}$$

Therefore, for all $n \geq N$, we have:

$$a_n > \frac{a+b}{2} > b_n$$

This contradicts our initial assumption that there exists $m_0 \in \mathbb{N} : \forall n \geq m_0, a_n \geq b_n$. Hence, our assumption that $a < b$ must be false. Therefore, we conclude that $a \geq b$. $\square$

Theorem 4.3.4 (Squeeze theorem). Let (a_n), (b_n), and (c_n) be three sequences such that $a_n \leq b_n \leq c_n$ for all n greater than some index N. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Proof. Let (a_n), (b_n), and (c_n) be three sequences such that $a_n \leq b_n \leq c_n$ for all n greater than some index N. We are given that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$. We want to show that $\lim_{n\to\infty} b_n = L$. By the definition of limits, for every $\epsilon > 0$, there exist natural numbers N_1 and N_2 such that for all $n \geq N_1$, $|a_n - L| < \epsilon$ and for all $n \geq N_2$, $|c_n - L| < \epsilon$. Let $N' = \max(N, N_1, N_2)$. Then, for all $n \geq N'$, we have:

$$|a_n - L| < \epsilon \implies L - \epsilon < a_n < L + \epsilon$$

and

$$|c_n - L| < \epsilon \implies L - \epsilon < c_n < L + \epsilon$$

Since $a_n \leq b_n \leq c_n$, it follows that:

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$$

Therefore, for all $n \geq N'$, we have:

$$|b_n - L| < \epsilon$$

This shows that for every $\epsilon > 0$, there exists a natural number N' such that for all $n \geq N'$, $|b_n - L| < \epsilon$. By the definition of limits, we conclude that $\lim_{n\to\infty} b_n = L$. $\square$

Example. Let's find the limit of the sequences defined by $(a_n) = \frac{5n^2+6n\cos(n)}{4n^2+3}$ and $(b_n) = \frac{\sqrt{3n^2+1}}{5n+2}$. We will use the squeeze theorem to find these limits. We have:

$$-1 \leq \cos(n) \leq 1 \implies 5n^2 - 6n \leq 5n^2 + 6n\cos(n) \leq 5n^2 + 6n$$

Thus:

$$\frac{5n^2-6n}{4n^2+3} \leq a_n \leq \frac{5n^2+6n}{4n^2+3}$$

We can find the limits of the sequences on the left and right:

$$\lim_{n\to\infty} \frac{5n^2-6n}{4n^2+3} = \lim_{n\to\infty} \frac{n^2}{n^2} \cdot \frac{5-\frac{6}{n}}{4+\frac{3}{n^2}} = \frac{5}{4}$$

$$\lim_{n\to\infty} \frac{5n^2+6n}{4n^2+3} = \lim_{n\to\infty} \frac{n^2}{n^2} \cdot \frac{5+\frac{6}{n}}{4+\frac{3}{n^2}} = \frac{5}{4}$$

By the squeeze theorem, we conclude that:

$$\lim_{n\to\infty} a_n = \frac{5}{4}$$

Now, let's find the limit of the sequence $(b_n) = \frac{\sqrt{3n^2+1}}{5n+2}$. We have:

$$\frac{\sqrt{3n^2+1}}{5n+2} = \frac{\sqrt{3n^2}\sqrt{1+\frac{1}{3n^2}}}{5n+2} = \frac{n\sqrt{3}\sqrt{1+\frac{1}{3n^2}}}{5n+2} = \frac{\sqrt{3}\sqrt{1+\frac{1}{3n^2}}}{5+\frac{2}{n}}$$

We can find bounds for b_n:

$$1 \leq \sqrt{1+\frac{1}{3n^2}} \leq \sqrt{1+\frac{1}{3n^2}+(\frac{1}{6n^2})^2} = \sqrt{(1+\frac{1}{6n^2})2} = 1+\frac{1}{6n^2}$$

Thus:

$$\frac{\sqrt{3}}{5+\frac{2}{n}} \leq b_n \leq \frac{\sqrt{3}(1+\frac{1}{6n^2})}{5+\frac{2}{n}}$$

We can find the limits of the sequences on the left and right:

$$\lim_{n\to\infty} \frac{\sqrt{3}}{5+\frac{2}{n}} = \frac{\sqrt{3}}{5}$$

$$\lim_{n\to\infty} \frac{\sqrt{3}(1+\frac{1}{6n^2})}{5+\frac{2}{n}} = \frac{\sqrt{3}}{5}$$

By the squeeze theorem, we conclude that:

$$\lim_{n\to\infty} b_n = \frac{\sqrt{3}}{5}$$

Example. Let's prove the limit of $a_n = \sqrt[n]{a},\ a > 0$ is 1.
Let $a > 1$ then $\forall x \in \mathbb{R} \implies (x-1)(x^{n-1}+x^{n-2}+\ldots+x+1) = x^n - 1,\ \forall n \in \mathbb{N}^*$. Thus, we have:

$$x - 1 = \frac{x^n - 1}{x^{n-1}+x^{n-2}+\ldots+x+1}$$

Let $x = \sqrt[n]{a} > 1$ then we have:

$$0 < \sqrt[n]{a} - 1 = \frac{a-1}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \ldots + \sqrt[n]{a} + 1} < \frac{a-1}{n}$$

By the squeeze theorem, we conclude that:

$$\lim_{n \to \infty} a_n = 1$$

If $a = 1$ then $a_n = 1$ thus $\lim_{n \to \infty} a_n = 1$.
If $0 < a < 1$ then we can write $a_n = \sqrt[n]{\frac{1}{b}}$ with $b = \frac{1}{a} > 1$. Thus, we have:

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \sqrt[n]{\frac{1}{b}} = \frac{1}{\lim_{n \to \infty} \sqrt[n]{b}} = \frac{1}{1} = 1$$

4.3.4 Geometric Sequences

Definition 4.3.4 (Geometric sequence).
A geometric sequence is a sequence of numbers where each term after the first is found by multiplying the previous term by a constant called the common ratio. The general form of a geometric sequence can be expressed as:

$$a_n = a_0 \cdot r^n$$

for $n \in \mathbb{N}$, $a_0 \in \mathbb{R}$, $a_0 \neq 0$ and $r \in \mathbb{R}$. We then have the following limites:

- If $|r| < 1$, then $\lim_{n \to \infty} a_n = 0$.
- If $r = 1$, then $\lim_{n \to \infty} a_n = a_0$.
- If $|r| > 1$ or $r = -1$, then the sequence diverges.

Proof. Let (a_n) be a geometric sequence defined by:

$$a_n = a_0 \cdot r^n$$

for $n \in \mathbb{N}$, $a_0 \in \mathbb{R}$, $a_0 \neq 0$ and $r \in \mathbb{R}$. We analyze the limit of the sequence (a_n) as n approaches infinity based on the value of the common ratio r:
1. If $r > 1$:
We can rewrite r as $r = 1 + x$ where $x > 0$. Then we have:

$$r^n = (1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n} x^n \geq 1 + \binom{n}{1} x = 1 + nx \quad \forall n \geq 1$$

Then by the Archimedean property, $\forall M > 0$, $\exists n \in \mathbb{N} : (1 + nx) > M$:

$$|a_0 r^n| = |a_0 (1+x)^n| \geq |a_0||1 + nx| > |a_0| M$$

Thus, the sequence is not bounded and diverges.
2. If $0 < r < 1$:

We can rewrite as $q = \frac{1}{r} > 1$ and we have $\forall M > 0$, $\exists n_0 \in \mathbb{N} : q^n > M$ for all $n \geq n_0$. Let $\epsilon > 0$ and $M = \frac{|a_0|}{\epsilon}$ we then have:

$$q^n > \frac{|a_0|}{\epsilon} \implies \frac{1}{q^n} = r^n < \frac{\epsilon}{|a_0|} \quad \forall n \geq n_0$$

Thus, we have:

$$|a_0 r^n - 0| < \epsilon \quad \forall n \geq n_0$$

By the definition of limits, we conclude that:

$$\lim_{n\to\infty} a_n = 0$$

3. If $r = 1$:
We have $a_0 r^n = a_0$ for all $n \in \mathbb{N}$. Thus, we have:

$$\lim_{n\to\infty} a_0 r^n = a_0$$

$\square$

Example. Let $a_n = (-3)^{-3n}$. Then we have:

$$a_n = ((-3)^{-3})^n = ((-1)^{-3} \cdot 3^{-3})^n = (-1 \cdot \frac{1}{27})^n = (-\frac{1}{27})^n$$

Then $r = -\frac{1}{27}$ thus $|r| < 1$. By the definition of geometric sequence, we conclude that:

$$\lim_{n\to\infty} a_n = 0$$

Example. Let $a_n = \sqrt{\frac{2^{2n+1}}{3^{n+8}}}$. Then we have:

$$a_n = \sqrt{\frac{2 \cdot 4^n}{3^8 \cdot 3^n}} = \frac{\sqrt{2}}{\sqrt{3^8}} \cdot \left(\frac{4}{3}\right)^{n/2} = \frac{\sqrt{2}}{81} \left(\sqrt{\frac{4}{3}}\right)^n$$

Then $r = \sqrt{\frac{4}{3}}$ thus $|r| > 1$. By the definition of geometric sequence, we conclude that the sequence diverges.

4.3.5 Properties of Limits

Some properties of limits:

- If $\lim_{n\to\infty} x_n = l \in \mathbb{R}$, then $\lim_{n\to\infty} |x_n| = |l|$.
- If $\lim_{n\to\infty} |x_n| = 0$, then $\lim_{n\to\infty} x_n = 0$.
- If $\lim_{n\to\infty} |x_n| = l \in \mathbb{R}^*$, then $\lim_{n\to\infty} x_n$ does not exist.
- If a_n is bounded and $b_n \to 0$, then $a_n b_n \to 0$.

Proof. Let's prove that if a_n is bounded and $b_n \to 0$, then $a_n b_n \to 0$.
Since (a_n) is bounded, there exists a constant $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Since (b_n) converges to 0, for every $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$, $|b_n - 0| < \frac{\epsilon}{M}$. This implies that:

$$|b_n| < \frac{\epsilon}{M}$$

Now, for all $n \geq N$, we have:

$$|a_n b_n| = |a_n| \cdot |b_n| \leq M \cdot |b_n| < M \cdot \frac{\epsilon}{M} = \epsilon$$

This shows that for every $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$, $|a_n b_n - 0| < \epsilon$. By the definition of limits, we conclude that $\lim_{n\to\infty} a_n b_n = 0$. □

4.3.6 Alembert's test

Theorem 4.3.5 (Alembert's test). Let (a_n) be a sequence such that $a_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \geq 0$. Then:

- If $L < 1$, then $\lim_{n\to\infty} a_n = 0$.
- If $L > 1$, then the sequence (a_n) diverges.
- If $L = 1$, the test is inconclusive.

Proof. Let $L < 1$ Then we have:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1 \implies \forall \epsilon > 0,\ \exists N \in \mathbb{N} : \forall n \geq N, \left|\frac{a_{n+1}}{a_n}\right| \leq L + \epsilon < 1$$

Let $m > N$, we have:

$$\underbrace{\left|\frac{a_m}{a_{m-1}}\right|}_{\leq L+\epsilon} \cdot \underbrace{\left|\frac{a_{m-1}}{a_{m-2}}\right|}_{\leq L+\epsilon} \cdot \ldots \cdot \underbrace{\left|\frac{a_{N+1}}{a_N}\right|}_{\leq L+\epsilon} = \left|\frac{a_m}{a_N}\right| \leq (L+\epsilon)^{m-N}$$

We then have:

$$0 \leq |a_m| \leq |a_N|(L+\epsilon)^{m-N} \to 0 \text{ (Geometric Sequence)}$$

By the squeeze theorem, we conclude that:

$$\lim_{m\to\infty} a_m = 0$$

Let $L > 1$. Then we have:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1 \implies \forall \epsilon > 0,\ \exists N \in \mathbb{N} : \forall n \geq N, \left|\frac{a_{n+1}}{a_n}\right| \geq L - \epsilon > 1$$

Let $m > N$, we have:

$$\underbrace{\left|\frac{a_m}{a_{m-1}}\right|}_{\geq L-\epsilon} \cdot \underbrace{\left|\frac{a_{m-1}}{a_{m-2}}\right|}_{\geq L-\epsilon} \cdot \ldots \cdot \underbrace{\left|\frac{a_{N+1}}{a_N}\right|}_{\geq L-\epsilon} = \left|\frac{a_m}{a_N}\right| \geq (L-\epsilon)^{m-N}$$

We then have:

$$|a_m| \geq |a_N|(L-\epsilon)^{m-N} \to \infty \text{ (Geometric Sequence)}$$

Thus, the sequence diverges. □

If $L = 1$ the test is inconclusive.

Example. Let $a_n = n$ then we have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n} = 1 + \frac{1}{n} \to 1$$

The sequence diverges.

Example. Let's compute $\lim_{n\to\infty} \frac{n^3}{5^n}$. By applying Alembert's test, we have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} = \frac{(n+1)^3}{5n^3} = \frac{1}{5}\left(1 + \frac{1}{n}\right)^3 \to \frac{1}{5} < 1$$

By the definition of Alembert's test, we conclude that:

$$\lim_{n\to\infty} \frac{n^3}{5^n} = 0$$

Example. Let's compute $\lim_{n\to\infty} \frac{5^n}{n!}$. By applying Alembert's test, we have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{5}{n+1} \to 0 < 1$$

By the definition of Alembert's test, we conclude that:

$$\lim_{n\to\infty} \frac{5^n}{n!} = 0$$

Note that $\lim_{n\to\infty} \frac{S^n}{n!} = 0$ since $n!$ grows faster than S^n for any constant S.

4.4 Infinite Limits

Definition 4.4.1 (Infinite limit).
A sequence (a_n) is said to diverge to infinity (or negative infinity) if for every real number $M > 0$ (or $M < 0$), there exists a natural number N such that for all $n \geq N$, $a_n > M$ (or $a_n < M$). In this case, we write:

$$\lim_{n\to\infty} a_n = +\infty \quad \text{or} \quad \lim_{n\to\infty} a_n = -\infty$$

Note that not all divergent sequences diverge to infinity. For example, the sequence defined by $a_n = (-1)^n$ oscillates between -1 and 1 and does not diverge to infinity or negative infinity.

4.4.1 Properties of infinite limits and indeterminate forms

- $\lim_{n\to\infty} a_n = \infty = \lim_{n\to\infty} b_n \implies \lim_{n\to\infty}(a_n + b_n) = \infty$
- $\lim_{n\to\infty} a_n = \pm\infty$ and b_n is bounded $\implies \lim_{n\to\infty}(a_n + b_n) = \pm\infty$
- $\lim_{n\to\infty} b_n = \infty$ and $a_n \geq b_n \quad (\forall n \geq n_0) \implies \lim_{n\to\infty} a_n = \infty$

- If a_n is bounded and $\lim_{n\to\infty} b_n = \pm\infty$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$
- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = +\infty$ and $a_n \neq 0 \quad \forall n \implies (a_n)$ does not converge

Some indeterminate forms are:

- $\infty - \infty$
- $0 \cdot \infty$
- $\frac{\pm\infty}{\pm\infty}$
- $\frac{0}{0}$

Example. Let $a_n = \sqrt{n} - \sqrt{n+2}$. Since if we take the limit directly we have an indeterminate form of type $\infty - \infty$, we can write:

$$a_n = \frac{(\sqrt{n} - \sqrt{n+2})(\sqrt{n} + \sqrt{n+2})}{\sqrt{n} + \sqrt{n+2}} = \frac{n - (n+2)}{\sqrt{n} + \sqrt{n+2}} = \frac{-2}{\sqrt{n} + \sqrt{n+2}}$$

We then have:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-2}{\sqrt{n} + \sqrt{n+2}} = \frac{-2}{\infty} = 0$$

Example. Let $a_n = n^2 - n$. Since if we take the limit directly we have an indeterminate form of type $\infty - \infty$, we can write:

$$a_n = n^2 - n = n^2 \left(1 - \frac{1}{n}\right)$$

We then have:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 \left(1 - \frac{1}{n}\right) = \infty \cdot 1 = \infty$$

Example. Let's compute $\lim_{n\to\infty} \frac{1+\cos(n)}{1+n} \cdot (n+1)$. Since if we take the limit directly we have an indeterminate form of type $0 \cdot \infty$, we can write:

$$a_n = \frac{1+\cos(n)}{1+n} \cdot (n+1) = 1 + \cos(n)$$

We then have:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 + \cos(n)$$

The sequence does not converge since $\cos(n)$ oscillates between -1 and 1.

Note that $(\cos(n))$ and $(\sin(n))$ are divergent sequences.

Proof. Let's suppose that $\exists \lim_{n\to\infty} (\cos(n)) = l \in \mathbb{R}$. Then we have:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} : \forall n \geq N, |\cos(n) - l| < \epsilon$$

Let $\epsilon = \frac{1}{2}$, then we have:

$$\forall n \geq N, l - \frac{1}{2} < \cos(n) < l + \frac{1}{2}$$

We then have:

$$\forall n \geq N, -1 \leq \cos(n) < l + \frac{1}{2} \implies l + \frac{1}{2} > -1 \implies l > -\frac{3}{2}$$

and

$$\forall n \geq N, l - \frac{1}{2} < \cos(n) \leq 1 \implies l - \frac{1}{2} < 1 \implies l < \frac{3}{2}$$

Thus, we have $-\frac{3}{2} < l < \frac{3}{2}$.
Let $k \in \mathbb{N}$ such that $k > N$ and $\cos(k) > l$. Such a k exists since $\cos(n)$ oscillates between -1 and 1. Then we have:

$$l < \cos(k) < l + \frac{1}{2}$$

Let $m \in \mathbb{N}$ such that $m > k$ and $\cos(m) < l$. Such a m exists since $\cos(n)$ oscillates between -1 and 1. Then we have:

$$l - \frac{1}{2} < \cos(m) < l$$

We then have:

$$|\cos(k) - l| < \frac{1}{2}$$

and

$$|\cos(m) - l| < \frac{1}{2}$$

But we also have:

$$|\cos(k) - \cos(m)| = |\cos(k) - l + l - \cos(m)| \leq |\cos(k) - l| + |l - \cos(m)| < \frac{1}{2} + \frac{1}{2} = 1$$

But we know that $|\cos(k) - \cos(m)|$ can be as large as 2 since $\cos(n)$ oscillates between -1 and 1. This is a contradiction. Thus, our initial assumption that $\lim_{n\to\infty}(\cos(n)) = l \in \mathbb{R}$ must be false. Therefore, we conclude that the sequence $(\cos(n))$ does not converge. □

Theorem 4.4.1. For all growing (decreasing) sequences that is upper (lower) bounded converges to a limit $l \in \mathbb{R}$ which is its supremum (infimum).
For all growing (decreasing) sequences that is not upper (lower) bounded diverges to $+\infty$ (respectively $-\infty$).
It's denoted by:

$$(a_n) \uparrow \quad \Leftrightarrow \quad (a_n) \text{ is growing}$$

$$(a_n) \downarrow \quad \Leftrightarrow \quad (a_n) \text{ is decreasing}$$

Proof. Let $(a_n) \uparrow$ and upper bounded. The we have:

$$\exists l = \sup\{a_n, n \in \mathbb{N}\}$$

By the definition of supremum, we have:

$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} : \forall n \geq N, |a_n - l| \leq \epsilon$$

Since (a_n) is growing, we have:

$$a_N \leq a_n \quad \forall n \geq N \implies l - a_n \leq l - a_N$$

Thus, we have:

$$0 \geq l - a_n \geq \epsilon \implies |l - a_n| \geq \epsilon \implies \lim_{n \to \infty} a_n = l$$

Let $(a_n) \uparrow$ not upper bounded. Then we have:

$$\forall A > 0,\ \exists a_{n_0} \geq A,\ (a_n) \uparrow\ \forall n \geq n_0 \implies a_n \geq a_{n_0} \geq A \implies \lim_{n \to \infty} a_n = +\infty$$

The proof for decreasing sequences is similar. □

4.5 The number e

Let $(x_n) : x_0 = 1, x_n = (1 + \frac{1}{n})^n$ for all $n \geq 1$ and let $(y_n) : y_0 = 1, y_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$ for all $n \geq 1$. We then have:

- $x_n \leq y_n$ for all $n \in \mathbb{N}$.
- $y_n \leq 3$ for all $n \in \mathbb{N}$.
- $(y_n) \uparrow$ for all $n \in \mathbb{N}$.
- $(x_n) \uparrow$ for all $n \in \mathbb{N}$.

Proof. **1.** Let's prove that $x_n \leq y_n$ for all $n \in \mathbb{N}$. We have:

$$x_n = (1 + \frac{1}{n})^n = 1 + (\binom{n}{1}\frac{1}{n}) + (\binom{n}{2}\frac{1}{n^2}) + \ldots + (\binom{n}{n}\frac{1}{n^n})$$

with:

$$\binom{n}{k}\left(\frac{1}{n}\right)^k = \frac{n!}{k!(n-k)!}\left(\frac{1}{n}\right)^k = \frac{1}{k!} \cdot \underbrace{\frac{n}{n}}_{\leq 1} \cdot \underbrace{\frac{n-1}{n}}_{\leq 1} \cdot \ldots \cdot \underbrace{\frac{n-k+1}{n}}_{\leq 1} \leq \frac{1}{k!}$$

Thus, we have:

$$x_n = 1 + (\binom{n}{1}\frac{1}{n}) + (\binom{n}{2}\frac{1}{n^2}) + \ldots + (\binom{n}{n}\frac{1}{n^n}) \leq 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} = y_n$$

for all $n \geq 1$.
2. Let's prove that $y_n \leq 3$ for all $n \in \mathbb{N}$. By recurrence, we have:

$$\frac{1}{k!} \leq \frac{1}{2^{k-1}} \quad \forall k \geq 1$$

Initialization: for $k = 1$, we have $1 \leq 1$ which is true.
Heredity: let's suppose that $\frac{1}{k!} \leq \frac{1}{2^{k-1}}$ for some $k \geq 1$. Then we have:

$$\frac{1}{(k+1)!} = \frac{1}{(k+1)k!} \leq \frac{1}{(k+1)2^{k-1}} \leq \frac{1}{2^k}$$

Thus, we have:

$$y_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} \leq 1 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n-1}} = 1 + \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} < 1\frac{1}{1 - \frac{1}{2}} = 1 + 2 = 3$$

for all $n \geq 1$.
3. Let's prove that $(y_n) \uparrow$ for all $n \in \mathbb{N}$. We have:

$$y_{n+1} = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \frac{1}{(n+1)!} = y_n + \frac{1}{(n+1)!} \geq y_n$$

for all $n \geq 1$.
4. Let's prove that $(x_n) \uparrow$ for all $n \in \mathbb{N}$. We want to show that:

$$\left(1 + \frac{1}{n+1}\right)^{n+1} \geq \left(1 + \frac{1}{n}\right)^n$$

In other words, we want to show that the geometric mean ($\sqrt{xy}$) is less than or equal to the arithmetic mean ($\frac{x+y}{2}$). More generally, we want to show that:

$$\sqrt[n+1]{a_0 a_1 \ldots a_n} \leq \frac{a_0 + a_1 + \ldots + a_n}{n+1}$$

Let $a_0 = 1$ and $a_n = \left(1 + \frac{1}{n}\right)$. Then we have:

$$\sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n} \leq \frac{1 + n\left(1 + \frac{1}{n}\right)}{n+1} \implies \sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n} \leq \frac{1 + n + 1}{n+1} = 1 + \frac{1}{n+1}$$

$$\implies \left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n+1}\right)^{n+1}$$

$\square$

Definition 4.5.1 (The number e).
The number e is defined as the limit of the sequence (x_n) or (y_n):

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right)$$

The number e is an irrational number approximately equal to 2.71828.

The following limits hold:

- $\lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n = e^{-x}, \ \forall x \in \mathbb{R}$
- $\lim_{n\to\infty} \frac{n!}{n^n} = 0$
- $\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1$
- $\lim_{n\to\infty} \sin(\frac{1}{n}) = 0$

4.6 Sequences defined by recurrence

Definition 4.6.1 (Linear recurrence).
A linear recurrence relation is an equation that defines a sequence of numbers based on a linear combination of its previous terms. The general form of a linear recurrence relation of order k can be expressed as:

$$a_{n+1} = q a_n + b$$

for $n \in \mathbb{N}$, $a_0 \in \mathbb{R}$ and $q, b \in \mathbb{R}$. We have the following:

- If $|q| \geq 1$ then the sequence diverges except if (a_n) is a constant sequence.
- If $|q| < 1$ then the sequence converges to $\frac{b}{1-q}$.

Proof. **1.** Let's prove that if $|q| > 1$ then the sequence diverges.
Let $|q| > 1$ and $a_0 = \frac{b}{1-q}$. Then we have:

$$a_1 = qa_0 + b = q \cdot \frac{b}{1-q} + b = \frac{b}{1-q}(q + 1 - q) = \frac{b}{1-q} = a_0$$

By recurrence, we conclude that (a_n) is a constant sequence.
Let $|q| > 1$ and $a_0 \neq \frac{b}{1-q}$. Then we have:

$$|a_{n+1} - l| = |q|^{n+1} \cdot |a_0 - l|$$

Since $|q| > 1$, we have $|q|^{n+1} \to \infty$. Thus, the sequence diverges.
If $q = 1$ then we have:

$$a_{n+1} = a_n + b$$

which is an arithmetic sequence that diverges except if $b = 0$ in which case (a_n) is a constant sequence.
If $q = -1$ then we have:

$$a_{n+1} = -a_n + b$$

which oscillates between two values.
2. Let's prove that if $|q| < 1$ then the sequence converges to $\frac{b}{1-q} = l$.
Let $|q| < 1$. Then we have for all $n \geq 1$:

$$0 \leq |a_{n+1} - l| = |qa_n + b - l| = |qa_n + b - (ql + b)| = |q| \cdot |a_n - l| = |q|^2 \cdot |a_{n-1} - l| = \ldots = |q|^{n+1} \cdot |a_0 - l|$$

Since $|q| < 1$, we have $|q|^{n+1} \to 0$. By the squeeze theorem, we conclude that:

$$\lim_{n \to \infty} a_n = l = \frac{b}{1-q}$$

□

Definition 4.6.2 (Non-linear recurrence).
A non-linear recurrence relation is an equation that defines a sequence of numbers based on a non-linear combination of its previous terms. The general form of a non-linear recurrence relation can be expressed as:

$$a_{n+1} = f(a_n)$$

for $n \in \mathbb{N}$, $a_0 \in \mathbb{R}$ and $f : E \to E \subset \mathbb{R}$. We have the following:

- $\exists m, M \in \mathbb{R} : m \leq g(x) \leq M \quad \forall x \in E$
- g is growing : $\forall x_1, x_2 \in E : x_1 \leq x_2 \Rightarrow g(x_1) \leq g(x_2)$

Then the sequence (a_n) is bounded and growing thus it converges to a limit $l \in \mathbb{R}$.

Note that if (2) is replaced by (2') g is decreasing : $\forall x_1, x_2 \in E : x_1 \leq x_2 \Rightarrow g(x_1) \geq g(x_2)$, then the sequence (a_n) is not bounded (but a_n might converge).

Example. Let $x_{n+1} = 5 - \frac{6}{x_n}$ and $x_0 = 4$. Then we have for the limit:

$$l = 5 - \frac{6}{l} \quad \Leftrightarrow \quad l^2 - 5l + 6 = 0 \quad \Leftrightarrow \quad l_1 = 2, l_2 = 3$$

We remark that $g(x) = 5 - \frac{6}{x}$ is a growing function for $x > 0$ then (x_n) is bounded. If we compute the first terms of the sequence, we have:

$$x_0 = 4$$

$$x_1 = 5 - \frac{6}{4} = \frac{7}{2} = 3.5 < x_0$$

$$x_2 = 5 - \frac{2}{7} = \frac{23}{7} < x_1$$

Thus (x_n) is a decreasing sequence. We want to find a lower bound. Let $x > 3$ we have:

$$g(x) = 5 - \underbrace{\frac{6}{x}}_{>3} < 2$$

Thus since $x_0 = 4 > 3$, $x_1 > 3$ and $x_n > 3$ implies that $g(x_n) = x_{n+1} > 3$. Thus, we have $x_n > 3$ for all $n \in \mathbb{N}$.

Some recommendations to study sequences defined by recurrence:

- Find candidates for the limits by solving the equation $l = f(l)$. If there's no solution, the sequence diverges.
- Study the monotonicity of f (make a graph if possible).
  Linear Recurrence: $x_{n+1} = qx_n + b$ then:
  - If $|q| < 1$ then (x_n) converges to $\frac{b}{1-q}$.
  - If $|q| \geq 1$ then (x_n) diverges except if $(x_0) = \frac{b}{1-q}$ (constant sequence).
  - If $|q| = 1$ then (x_n) diverges except if (x_n) is constant.

  Non-linear Recurrence: $x_{n+1} = f(x_n)$ with $f(x)$ growing implies that (x_n) is bounded.
  - If $x_0 < x_1$ then $(x_n) \uparrow$, we need to show that (x_n) is upper bounded.
  - If $x_0 > x_1$ then $(x_n) \downarrow$, we need to show that (x_n) is lower bounded.

  In both cases (x_n) converges to a limit $l \in \mathbb{R}$.
  Proposition: If (x_n) and (a_n) two sequences such that $0 < a_n < 1$ for all $n \in \mathbb{N}$ and $\exists l \in \mathbb{R} : (x_{n+1} - l) = a_n(x_n - l)$. Then (x_n) converges.

4.7 Subsequences

Definition 4.7.1 (Subsequence).
A subsequence of a sequence (a_n) is a new sequence formed by selecting specific elements from the original sequence while maintaining their order. Formally, if (a_n) is a sequence and (n_k) is a strictly increasing sequence of natural numbers, then the subsequence (a_{n_k})

is defined as:

$$(a_{n_1}, a_{n_2}, a_{n_3}, \dots)$$

where $n_1 < n_2 < n_3 < \dots$

Example. Let $a_n = (-1)^n$. Then we have two subsequences:

$$a_{2n} = 1 \quad \text{and} \quad a_{2n+1} = -1$$

We can write that $a_{2n} \subset a_n$ and $a_{2n+1} \subset a_n$. Both subsequences converge but the original sequence diverges.

Theorem 4.7.1. If a sequence (a_n) converges to a limit $l \in \mathbb{R}$ then all its subsequences converge to the same limit l.

Proof. Let (a_n) be a sequence that converges to a limit $l \in \mathbb{R}$ and let (a_{n_k}) be a subsequence of (a_n). We have:

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} : \forall n \geq N, |a_n - l| < \epsilon$$

Since (n_k) is a strictly increasing sequence of natural numbers, we have:

$$n_k \geq k \quad \forall k \in \mathbb{N}$$

Let $k > N$, then we have:

$$n_k > N \implies |a_{n_k} - l| < \epsilon$$

Thus, we conclude that:

$$\lim_{k \to \infty} a_{n_k} = l$$

□

4.7.1 Bolzano-Weierstrass Theorem

Theorem 4.7.2 (Bolzano-Weierstrass Theorem). Every bounded sequence of real numbers has a convergent subsequence.

4.7.2 Cauchy subsequence

Definition 4.7.2 (Cauchy subsequence).
A subsequence (a_{n_k}) of a sequence (a_n) is called a Cauchy subsequence if for every $\epsilon > 0$, there exists a natural number N such that for all $m, n \geq N$, the following condition holds:

$$|a_{n_m} - a_{n_n}| < \epsilon$$

Theorem 4.7.3. A sequence of real numbers is Cauchy if and only if it converges.

If we have $\lim_{n\to\infty}(a_{n+k} - a_n) = 0$ for all $k \in \mathbb{N}$, this does not imply that (a_n) is Cauchy.

Example. Let $a_n = \sqrt{n}$. Then we have:

$$a_{n+k} - a_n = \sqrt{n+k} - \sqrt{n} = \frac{(\sqrt{n+k} - \sqrt{n})(\sqrt{n+k} + \sqrt{n})}{\sqrt{n+k} + \sqrt{n}} = \frac{k}{\sqrt{n+k} + \sqrt{n}} \to 0$$

But (a_n) is divergent which implies that it is not Cauchy.

4.8 Upper limit and lower limit of a sequence

Definition 4.8.1 (Upper/lower limit).
Let (a_n) be a bounded sequence of real numbers. Two sequence can be defined as follows:

$$y_n = \sup\{a_k : k \geq n\}$$

$$z_n = \inf\{a_k : k \geq n\}$$

The sequence (y_n) is called the sequence of upper bounds, and the sequence (z_n) is called the sequence of lower bounds. Both sequences are monotonic, with (y_n) being decreasing and (z_n) being increasing. Since both sequences are bounded, they converge to limits L and l respectively. These limits are called the upper limit (or limit superior) and lower limit (or limit inferior) of the sequence (a_n):

$$\limsup_{n\to\infty} a_n = L = \lim_{n\to\infty} y_n$$

$$\liminf_{n\to\infty} a_n = l = \lim_{n\to\infty} z_n$$

It is always true that $l \leq L$. If $l = L$, then the sequence (a_n) converges to the common limit $l = L$.

Example. Let $a_n = (-1)^n \cdot (1 + \frac{1}{n^2})$ for all $n \in \mathbb{N}$. We can easily see that this sequence is bounded since:

$$-2 < a_n < 2 \quad \forall n \in \mathbb{N}$$

We then have:

$$a_n = (-2, 1 + \frac{1}{4}, -1 - \frac{1}{9}, 1 + \frac{1}{16}, -1 - \frac{1}{25}, \ldots)$$

for $n = 1, 2, 3, 4, 5, \ldots$. We then have:

$$y_n = (1 + \frac{1}{4}, 1 + \frac{1}{4}, 1 + \frac{1}{16}, 1 + \frac{1}{16}, \ldots, 1 + \frac{1}{(n + \frac{1+(-1)^{n+1}2}{)})^2} \quad \text{and} \quad \lim_{n\to\infty} y_n = 1$$

$$z_n = \inf\{a_k : k \geq n\} = -1 - \frac{1}{(n+1)^2}$$

If $\limsup x_n = l$ and $\liminf x_n = l_2$, then there exists two subsequences (x_{n_k}) and (x_{m_k}) such that:

$$\lim_{k\to\infty} x_{n_k} = l_1 \quad \text{and} \quad \lim_{k\to\infty} x_{m_k} = l_2$$

If $\limsup x_n = \liminf x_n = l$ then $\lim x_n = l$.
Remark also that if $\lim x_n = l$ then $\limsup x_n = \liminf x_n = l$.

Example. Let $a_n = (-1)^n(1 + \frac{1}{n^2})$. We then have:

$$\limsup a_n = 1 \quad \text{and} \quad \liminf a_n = -1$$

Thus (a_n) diverges.

Example. Let $a_n = 1 - \frac{1}{n}$. We then have:

$$x_n = (0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots)$$

We then have:

$$y_n = \sup\{a_k : k \geq n\} = 1 - \frac{1}{n} \quad \text{and} \quad \lim_{n\to\infty} y_n = 1$$

Thus:

$$\limsup a_n = 1 \quad \text{and} \quad \liminf a_n = 1$$

Thus (a_n) converges to 1.

Chapter 5

Numerical Series

Definition 5.0.1 (Series).
Let $(a_n)_{n\geq 0}$ be a sequence of real numbers. The series with general term a_n is the sequence defined by:

$$S_n = \sum_{k=0}^{n} a_k$$

The sequence $(S_n)_{n\geq 0}$ is called the sequence of partial sums of the series.

Definition 5.0.2 (Convergence of a Series).
A series $\sum_{n=0}^{\infty} a_n$ is said to converge if the sequence of its partial sums $(S_n)_{n\geq 0}$ converges. In this case, we denote its limit by:

$$S = \sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} S_n$$

If the series does not converge, it is said to diverge.

Example. Let $\sum_{n=0}^{\infty} n$. Then the sequence of partial sums is given by:

$$S_n = \sum_{k=0}^{n} k = 0 + 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

As $n \to \infty$, $S_n \to \infty$. Therefore, the series $\sum_{n=0}^{\infty} n$ diverges.

Example. Let $\sum_{k=0}^{\infty} \frac{1}{2^k}$ be a geometric series with first term $a = 1$ and common ratio $r = \frac{1}{2}$. Then the sequence of partial sums is given by:

$$S_n = \sum_{k=0}^{n} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^{n+1}}\right)$$

As $n \to \infty$, $S_n \to 2$. Therefore, the series $\sum_{k=0}^{\infty} \frac{1}{2^k}$ converges to 2.

Similarly, the series $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ for $|r| < 1$.

Example. Let $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ be the harmonic series. The sequence of partial sums is given by:

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

Let's show that S_n diverges as $n \to \infty$. By contradiction, assume that S_n converges to some limit L. Then we have:

$$S_{2n} = 1 + \frac{1}{2} + \ldots + \frac{1}{2n} = S_n + \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right)$$

Note that:

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \geq n \cdot \frac{1}{2n} = \frac{1}{2}$$

Thus, we have:

$$S_{2n} \geq S_n + \frac{1}{2}$$

Taking the limit as $n \to \infty$, we get:

$$L \geq L + \frac{1}{2}$$

which is a contradiction. Therefore, the harmonic series diverges.

Example. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. We can show that with:

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \underbrace{\left(\frac{1}{2^2} + \frac{1}{3^2}\right)}_{<2\cdot\frac{1}{2^2}} + \underbrace{\left(\frac{1}{4^2} + \frac{1}{5^2}\right)}_{<2\cdot\frac{1}{4^2}} + \underbrace{\left(\frac{1}{6^2} + \frac{1}{7^2}\right)}_{<2\cdot\frac{1}{6^2}} + \ldots + \frac{1}{n^2}$$

We then have:

$$S_n < 1 + 2\sum_{k=1}^{n} \frac{1}{(2k)^2} = 1 + 2\sum_{k=1}^{n} \frac{1}{4} \cdot \frac{1}{k^2} = 1 + \frac{1}{2}\underbrace{\sum_{k=1}^{n} \frac{1}{k^2}}_{=S_n}$$

Thus, we get:

$$S_n < 1 + \frac{1}{2}S_n \implies \frac{1}{2}S_n < 1 \implies S_n < 2$$

Therefore, the sequence of partial sums $(S_n)_{n\geq 1}$ is bounded above by 2, we can also show that it is increasing:

$$S_{n+1} = S_n + \frac{1}{(n+1)^2} > S_n$$

making $(S_n)_{n\geq 1}$ a monotone increasing and bounded sequence thus it converges. Hence, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Remark that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for all $p > 1$.

Definition 5.0.3 (Absolute Convergence).
A series $\sum_{n=0}^{\infty} a_n$ is said to converge absolutely if the series of absolute values $\sum_{n=0}^{\infty} |a_n|$ converges.

Theorem 5.0.1. If a series $\sum_{n=0}^{\infty} a_n$ converges absolutely, then it converges.

Definition 5.0.4 (Necessary Condition for Convergence).
If a series $\sum_{n=0}^{\infty} a_n$ converges, then the general term a_n tends to 0 as n tends to infinity:

$$\lim_{n\to\infty} a_n = 0$$

Proof. Let $S_n = \sum_{k=0}^{n} a_k$ be the sequence of partial sums of the series and it is Cauchy since it converges. Thus, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$:

$$|S_m - S_n| < \varepsilon$$

Without loss of generality, assume $m > n$. Then we have:

$$|a_{n+1} + a_{n+2} + \ldots + a_m| < \varepsilon$$

In particular, for $m = n + 1$, we get:

$$|a_{n+1}| < \varepsilon$$

Thus, as $n \to \infty$, $a_n \to 0$. □

Example. Let $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}$ be a series. The general term $\frac{(-1)^n}{2}$ does not tends to 0 as n tends to infinity. Therefore, by the necessary condition for convergence, the series diverges.

Remark that the necessary condition for convergence is not sufficient. For example, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ has general term tending to 0 but diverges.

5.1 Tests for Convergence

5.1.1 Leibniz Test for Alternating Series

Theorem 5.1.1 (Leibniz Test). Let $\sum_{n=0}^{\infty} a_n$ be a series such that:

- there exists $p \in \mathbb{N} : \forall n \geq p \implies |a_{n+1}| \leq |a_n|$,
- there exists $q \in \mathbb{N} : \forall n \geq q \implies a_{n+1} \cdot a_n \leq 0$ (the terms alternate in sign),
- $\lim_{n\to\infty} a_n = 0$,

Then the series $\sum_{n=0}^{\infty} a_n$ converges.

Example. Let $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ be the alternating harmonic series. We have:

- for all $n \geq 1$, $|a_{n+1}| = \frac{1}{n+1} \leq \frac{1}{n} = |a_n|$,
- for all $n \geq 1$, $a_{n+1} \cdot a_n = (-1)^{n+1}\frac{1}{n+1} \cdot (-1)^n\frac{1}{n} = -\frac{1}{n(n+1)} \leq 0$,
- $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n\frac{1}{n} = 0$,

Therefore, by the Leibniz test, the series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ converges.

5.1.2 Comparison Test

Theorem 5.1.2 (Comparison Test). Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two series with non-negative terms such that there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \leq b_n$. Then:

- if $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges,
- if $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} b_n$ diverges.

Proof. Assume that $\sum_{n=0}^{\infty} b_n$ converges. Let $S_n^a = \sum_{k=0}^{n} a_k$ and $S_n^b = \sum_{k=0}^{n} b_k$ be the sequences of partial sums of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, respectively. Since $a_n \leq b_n$ for all $n \geq N$, we have:

$$S_n^a = \sum_{k=0}^{n} a_k \leq \sum_{k=0}^{n} b_k = S_n^b$$

for all $n \geq N$. Since $\sum_{n=0}^{\infty} b_n$ converges, the sequence $(S_n^b)_{n\geq 0}$ is bounded. Therefore, the sequence $(S_n^a)_{n\geq 0}$ is also bounded. Moreover, since $a_n \geq 0$, the sequence $(S_n^a)_{n\geq 0}$ is increasing. Thus, by the Monotone Convergence Theorem, the sequence $(S_n^a)_{n\geq 0}$ converges, and hence the series $\sum_{n=0}^{\infty} a_n$ converges.
The second part of the theorem follows similarly by contraposition. $\square$

Example. Let $\sum_{n=0}^{\infty} \frac{\cos n!}{(n+1)^2}$ be a series. Let's consider the absolute convergence of this series. We have:

$$\left|\frac{\cos n!}{(n+1)^2}\right| \leq \frac{1}{(n+1)^2}$$

for all $n \geq 0$. Since the series $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$ converges, by the Comparison Test, the series $\sum_{n=0}^{\infty} \left|\frac{\cos n!}{(n+1)^2}\right|$ converges. Therefore, the series $\sum_{n=0}^{\infty} \frac{\cos n!}{(n+1)^2}$ converges absolutely, and hence it converges.

Remark that if $\sum_{n=0}^{\infty} a_n$ has only positive (negative) terms, and the sequence of the partial sums $(S_n)_{n\geq 0}$ is bounded above (below), then the series converges.

5.1.3 Alembert's Ratio Test

Theorem 5.1.3 (Alembert's Ratio Test). Let $\sum_{n=0}^{\infty} a_n$ be a series with positive terms. Suppose that the limit:

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists. Then:

- if $L < 1$, the series $\sum_{n=0}^{\infty} a_n$ converges,
- if $L > 1$, the series $\sum_{n=0}^{\infty} a_n$ diverges,
- if $L = 1$, the test is inconclusive.

5.1.4 Cauchy's Root Test

Theorem 5.1.4 (Cauchy's Root Test). Let a_n be a sequence and there exists the limit:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \in \mathbb{R}$$

or more generally:

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = L \in \mathbb{R}$$

Then:

- if $L < 1$, the series $\sum_{n=0}^{\infty} a_n$ converges absolutely,
- if $L > 1$, the series $\sum_{n=0}^{\infty} a_n$ diverges,
- if $L = 1$, the test is inconclusive.

The proof of this test involves using geometric series since we compare $|a_n|$ with L^n for large n. Remark that:

- if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$ and $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = l$, then $r = l$,
- sometimes $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists while $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist (and vice versa),
- if both limits are inconclusive (equal to 1), then no conclusion can be drawn about the series' convergence.

Example. Let $\sum_{n=1}^{\infty} n = \infty$ be a divergent series but:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent but:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1$$

Example. Let $\sum_{n=1}^{\infty} \frac{10^n \cdot n!}{(2n)!}$ be a series. We have:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{10^{n+1} \cdot (n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{10^n \cdot n!} = \lim_{n\to\infty} \frac{10(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{10}{4n+2} = 0$$

Since $0 < 1$, by Alembert's Ratio Test, the series $\sum_{n=1}^{\infty} \frac{10^n \cdot n!}{(2n)!}$ converges.

Example. Let $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ be a series. We have:

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{\frac{|x|^{k+1}}{(k+1)!}}{\frac{|x|^k}{k!}} = \lim_{k\to\infty} \frac{|x|}{k+1} = 0$$

Since $0 < 1$, by Alembert's Ratio Test, the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for all $x \in \mathbb{R}$. Note that this series defines the exponential function e^x for all $x \in \mathbb{R}$.

Chapter 6

Function of Real Numbers

6.1 Functions

Definition 6.1.1 (Real Function).
A function $f : E \to F$, where E and F are subsets of $\mathbb{R}$, is a rule that assigns to each element $x \in E$ a unique element $f(x) \in F$. The set $E = \mathrm{D}(f)$ is called the domain of the function, and the set F is called the codomain.

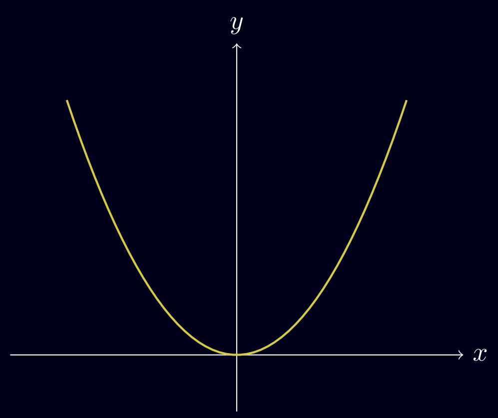
Definition 6.1.2 (Graph of a Function).
The graph of a function $f : E \to F$ is the set of points in the Cartesian plane defined by:

$$G(f) = \{(x, f(x)) \mid x \in E\}$$

Example. The graph of the function $f(x) = x^2$ is the set of points:

$$G(f) = \{(x, x^2) \mid x \in \mathbb{R}\}$$

Thus:



6.1.1 Properties of Functions

Definition 6.1.3 (Increasing and Decreasing Functions).
A function $f : E \to \mathbb{R}$ is said to be increasing on an interval $I \subseteq E$ if for all $x_1, x_2 \in I$ such that $x_1 < x_2$, we have:

$$f(x_1) \leq f(x_2)$$

If the inequality is strict, i.e., $f(x_1) < f(x_2)$, then f is said to be strictly increasing on I.
Similarly, a function $f : E \to \mathbb{R}$ is said to be decreasing on an interval $I \subseteq E$ if for all $x_1, x_2 \in I$ such that $x_1 < x_2$, we have:

$$f(x_1) \geq f(x_2)$$

If the inequality is strict, i.e., $f(x_1) > f(x_2)$, then f is said to be strictly decreasing on I.

Note that a function (strictly) increasing or (strictly) decreasing on an interval I is also called (strictly) monotonic on I.

Definition 6.1.4 (Even and Odd Functions).
A function $f : E \to \mathbb{R}$ is said to be even if for all $x \in E$, we have:

$$f(-x) = f(x)$$

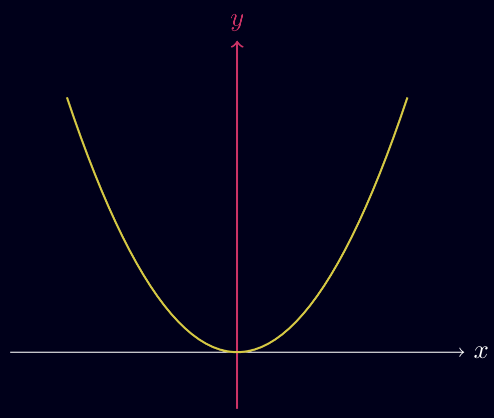
A function $f : E \to \mathbb{R}$ is said to be odd if for all $x \in E$, we have:

$$f(-x) = -f(x)$$

Example. The function $f(x) = x^2$ is even since for all $x \in \mathbb{R}$, we have:

$$f(-x) = (-x)^2 = x^2 = f(x)$$

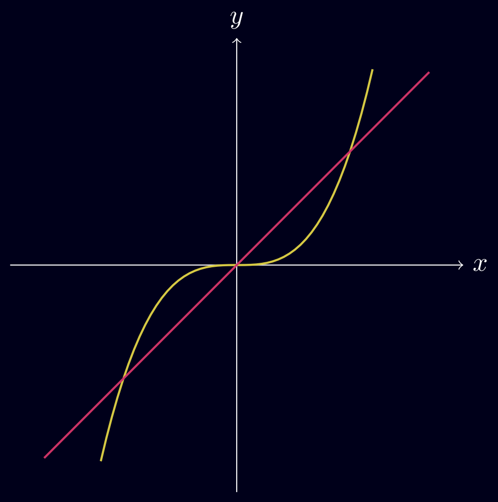
Graphically, even functions are symmetric with respect to the y-axis.



The function $g(x) = x^3$ is odd since for all $x \in \mathbb{R}$, we have:

$$g(-x) = (-x)^3 = -x^3 = -g(x)$$

Graphically, odd functions are symmetric with respect to the origin.



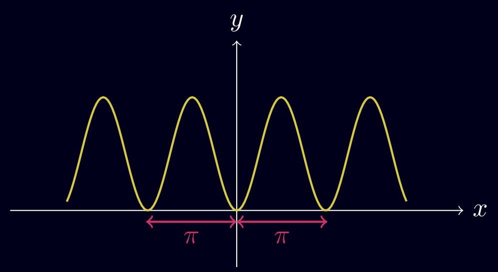
Definition 6.1.5 (Periodic Function).
A function $f : E \to \mathbb{R}$ is said to be periodic with period $T > 0$ if for all $x \in E$ such that $x \pm T \in E$, we have:

$$f(x \pm T) = f(x)$$

Example. The function $f(x) = \sin(x)^2$ is periodic with period 2π, since we have:

$$\sin(x)^2 = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 = \frac{e^{2ix} - 2 + e^{-2ix}}{-4} = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

Since $\cos(x)$ is periodic with period 2π, we have that $\sin(x)^2$ is also periodic with a period of π (or $\sin(x)^2$ is π-periodic). Graphically, we have:



Example. Some functions are periodic but it's impossible to find the smallest period. For example, the function:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

In other words, for any rational number P:

$$\begin{cases} \text{rational} + P = \text{rational} \\ \text{irrational} + P = \text{irrational} \end{cases}$$

Thus, $f(x+P) = f(x)$ for any rational number P, making f a periodic function without a smallest period.

6.1.2 Boundedness and Extrema of Functions

Definition 6.1.6 (Bounded Function).
A function $f : E \to \mathbb{R}$ is said to be bounded above on a set $A \subseteq E$ if the set $f(A) \subset \mathbb{R}$ is bounded above, i.e., there exists a real number M such that for all $x \in A$, we have:

$$f(x) \leq M$$

Similarly, f is said to be bounded below on A if the set $f(A) \subset \mathbb{R}$ is bounded below, i.e., there exists a real number m such that for all $x \in A$, we have:

$$f(x) \geq m$$

If f is both bounded above and bounded below on A, then it is said to be bounded on A.

Definition 6.1.7 (Supremum and Infimum).
A function $f : E \to \mathbb{R}$ has a supremum (least upper bound) on a set $A \subseteq E$, denoted by $\sup_{x \in A} f(x)$, if the set $f(A) \subset \mathbb{R}$ has a supremum. This means that:

$$\sup_{x \in A} f(x) = \sup\{f(x), x \in A\}$$

Similarly, f has an infimum (greatest lower bound) on A, denoted by $\inf_{x \in A} f(x)$, if the set $f(A) \subset \mathbb{R}$ has an infimum. This means that:

$$\inf_{x \in A} f(x) = \inf\{f(x), x \in A\}$$

Example. Let $f : (0,1) \to \mathbb{R}$ defined by $f(x) = x^2 + 3$ be bounded on $A = (0,1)$. Then:

$$\sup_{x \in A} f(x) = \sup\{x^2 + 3, x \in A\} = 4 \notin f(A)$$

Similarly:

$$\inf_{x \in A} f(x) = \inf\{x^2 + 3, x \in A\} = 3 \notin f(A)$$

Definition 6.1.8 (Local Maximum and Minimum).
A function $f : E \to \mathbb{R}$ has a local maximum at a point $x_0 \in E$ if there exists $\delta > 0$ such

that for all $x \in E$ with $|x - x_0| < \delta$, we have:

$$f(x) \leq f(x_0)$$

Similarly, f has a local minimum at a point $x_0 \in E$ if there exists $\delta > 0$ such that for all $x \in E$ with $|x - x_0| < \delta$, we have:

$$f(x) \geq f(x_0)$$

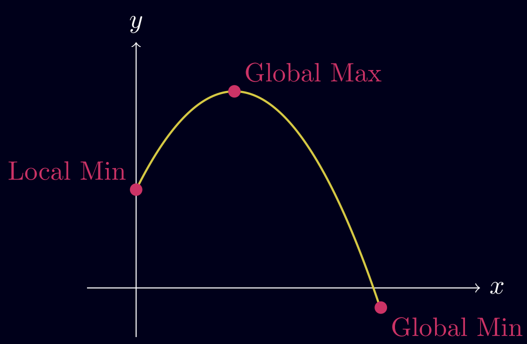
Definition 6.1.9 (Global Maximum and Minimum).
A function $f : E \to \mathbb{R}$ has a global maximum at a point $x_0 \in E$ if for all $x \in E$, we have:

$$f(x) \leq f(x_0)$$

Similarly, f has a global minimum at a point $x_0 \in E$ if for all $x \in E$, we have:

$$f(x) \geq f(x_0)$$

Example. Let's show the difference between local and global extrema graphically:



If the $\max_{x \in E} f(x)$ (or $\min_{x \in E} f(x)$) exists, then f is bounded above (or below) on E and $\sup_{x \in E} f(x) = \max_{x \in E} f(x)$ (or $\inf_{x \in E} f(x) = \min_{x \in E} f(x)$).
A bounded function on E does not necessarily reach its bounds, i.e., the maximum or minimum may not exist.

Example. Let $f : [0, 1) \to \mathbb{R}$ defined by $f(x) = x^2 + 3$. We clearly see that f is bounded but:

$$\max_{x \in [0,1)} f(x) = 4 \notin f([0, 1))$$

i.e. f does not reach its upper bound and:

$$\min_{x \in [0,1)} f(x) = 3 \in f([0, 1))$$

i.e. f reaches its lower bound at $x = 0$:

$$f(0) = 3 = \min_{x \in [0,1)} f = \inf_{x \in [0,1)} f$$

6.1.3 Types of Functions

Definition 6.1.10 (Surjectivity).
A function $f : E \to F$ is said to be surjective (onto) if for every $y \in F$, there exists at least one $x \in E$ such that $f(x) = y$.

Definition 6.1.11 (Injectivity).
A function $f : E \to F$ is said to be injective (one-to-one) if for every $x_1, x_2 \in E$, whenever $f(x_1) = f(x_2)$, it follows that $x_1 = x_2$ (i.e. there exists at most one $x \in E$ for each $y \in F$ such that $f(x) = y$).

If $f : E \to F$ is not injective, it can be made injective by restricting its domain E and if f is not surjective, it can be made surjective by adjusting its codomain F.

Definition 6.1.12 (Bijectivity).
A function $f : E \to F$ is said to be bijective if it is both injective and surjective. In this case, for every $y \in F$, there exists a unique $x \in E$ such that $f(x) = y$.

Definition 6.1.13 (Inverse Function).
Let $f : E \to F$ be a bijective function. The inverse function of f, denoted by $f^{-1} : F \to E$, is defined by:

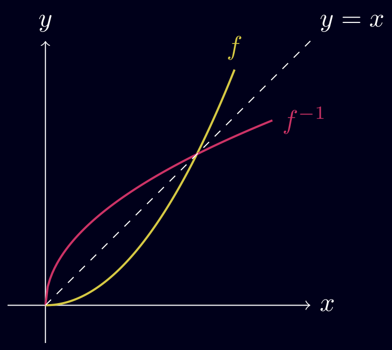
$$f^{-1}(y) = x \quad \text{where} \quad f(x) = y$$

for every $y \in F$.

By convention, for the trigonometric functions, the inverse sine, cosine, and tangent functions are defined on restricted domains to ensure bijectivity:

- $\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$, reciprocally $\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\cos : [0, \pi] \to [-1, 1]$, reciprocally $\arccos : [-1, 1] \to [0, \pi]$
- $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, reciprocally $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $\cot : (0, \pi) \to \mathbb{R}$, reciprocally $\operatorname{arccot} : \mathbb{R} \to (0, \pi)$

Example. Graphically, the inverse function f^{-1} of a bijective function f can be obtained by reflecting the graph of f across the line $y = x$:



Example. Let $f = \frac{1}{\cos(x)^2+1}$ defined on $\mathbb{R}$. Let's find the biggest interval containing $x = 1$ where f is bijective.
We have:

$$\frac{1}{y+1} \quad \text{is injective} \quad \forall y \in [0, +\infty)$$

and

$$\cos(x)^2 \quad \text{is injective} \quad \forall x \in [0, \frac{\pi}{2}]$$

Since $1 \in [0, \frac{\pi}{2}]$, thus f is injective on the interval $\left[0, \frac{\pi}{2}\right]$. We also have:

$$f(x) = \frac{1}{\cos(x)^2 + 1} \quad x \in \left[0, \frac{\pi}{2}\right]$$

is increasing since $\cos(x)^2$ is decreasing on $\left[0, \frac{\pi}{2}\right]$ and thus:

$$\inf_{x \in \left[0, \frac{\pi}{2}\right]} f(x) = f\left(\frac{\pi}{2}\right) = \frac{1}{2} \quad \text{and} \quad \sup_{x \in \left[0, \frac{\pi}{2}\right]} f(x) = f(0) = 1$$

Therefore, $f : \left[0, \frac{\pi}{2}\right] \to \left[\frac{1}{2}, 1\right]$ is bijective and its inverse function is given by:

$$f^{-1}(y) = \arccos\left(\sqrt{\frac{1}{y} - 1}\right) \quad y \in \left[\frac{1}{2}, 1\right]$$

6.1.4 Composite Functions

Definition 6.1.14 (Composite Function).
Let $f : E \to F$ and $g : F \to G$ be two functions. The composite function of f and g, denoted by $g \circ f : E \to G$, is defined by:

$$(g \circ f)(x) = g(f(x))$$

for every $x \in E$.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 3$ and let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$. Then, the composite function $g \circ f : \mathbb{R} \to \mathbb{R}$ is given by:

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = (2x + 3)^2 = 4x^2 + 12x + 9$$

Note that the order of composition matters, as $f \circ g$ would yield a different result:

$$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2 + 3$$

Example. Let $f : E \to F$ be a bijective function with inverse $f^{-1} : F \to E$. Then, the composite functions $f \circ f^{-1} : F \to F$ and $f^{-1} \circ f : E \to E$ are given by:

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = y \quad \forall y \in F$$

and

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x \quad \forall x \in E$$

Thus, composing a function with its inverse yields the identity function on the respective domains.

6.2 Limits of Functions

Definition 6.2.1 (Neighborhood).
A function $f : E \to F$ is defined on a neighborhood of a point $x_0 \in E$ if there exists $\delta > 0$ such that:

$$\{x \in E : 0 < |x - x_0| < \delta\} \subseteq E$$

Remark that the function does not need to be defined at x_0 itself, only in its vicinity.

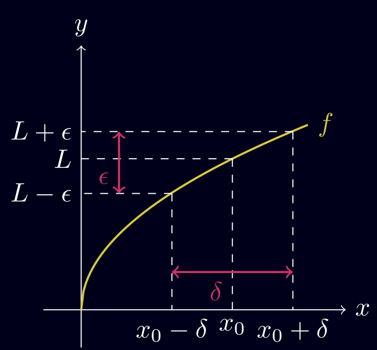
Example. Let $f = \frac{\sin(x)}{x}$ is defined on a neighborhood of $x_0 = 0$ but not at $x_0 = 0$ itself since $f(0)$ is undefined.

Definition 6.2.2 (Limit of a Function at a Point).
Let $f : E \to F$ be a function defined on a neighborhood of a point $x_0 \in E$. It is said that the limit of $f(x)$ as x approaches x_0 is equal to $L \in F$, denoted by:

$$\lim_{x \to x_0} f(x) = L$$

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in E$ with $0 < |x - x_0| < \delta$, we have:

$$|f(x) - L| < \epsilon$$



Example. Let $f(x) = 3x - 1$ and $x_0 = 1$. Then:

$$\lim_{x \to 1} f(x) = \lim_{x \to 1}(3x - 1) = 2$$

To show this using the ϵ-δ definition, let $\epsilon > 0$. We need to find $\delta > 0$ such that for all x with $0 < |x - 1| < \delta$, we have:

$$|f(x) - 2| < \epsilon$$

We have:

$$|f(x) - 2| = |3x - 1 - 2| = |3x - 3| = 3|x - 1|$$

Thus, we want:

$$3|x - 1| < \epsilon \quad \implies \quad |x - 1| < \frac{\epsilon}{3}$$

Therefore, we can choose $\delta = \frac{\epsilon}{3}$. Hence, for all x with $0 < |x - 1| < \delta$, we have:

$$|f(x) - 2| < \epsilon$$

This confirms that:

$$\lim_{x \to 1} f(x) = 2$$

Example. Let $f(x) = \sqrt{x} = \sqrt{x_0}, \forall x_0 > 0$. Then:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$$

To show this using the ϵ-δ definition, let $\epsilon > 0$. We need to find $\delta > 0$ such that for all x with $0 < |x - x_0| < \delta$, we have:

$$|\sqrt{x} - \sqrt{x_0}| < \epsilon$$

We have:

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|\sqrt{x} - \sqrt{x_0}| \cdot |\sqrt{x} + \sqrt{x_0}|}{|\sqrt{x} + \sqrt{x_0}|} = \frac{|x - x_0|}{|\sqrt{x} + \sqrt{x_0}|}$$

To ensure that $|\sqrt{x} - \sqrt{x_0}| < \epsilon$, we need:

$$\frac{|x - x_0|}{|\sqrt{x} + \sqrt{x_0}|} < \epsilon \quad \implies \quad |x - x_0| < \epsilon|\sqrt{x} + \sqrt{x_0}|$$

Since we are considering x in a neighborhood of x_0, we can assume that x is close to x_0. Thus, we can find a lower bound for $|\sqrt{x} + \sqrt{x_0}|$. For instance, if we restrict x to be in the interval $[x_0 - 1, x_0 + 1]$ (assuming $x_0 > 1$ for simplicity), we have:

$$|\sqrt{x} + \sqrt{x_0}| \geq \sqrt{x_0} \quad \text{for all } x \in [x_0 - 1, x_0 + 1]$$

Therefore, we can choose $\delta = \epsilon\sqrt{x_0}$. Hence, for all x with $0 < |x - x_0| < \delta$, we have:

$$|\sqrt{x} - \sqrt{x_0}| < \epsilon$$

This confirms that:

$$\lim_{x \to x_0} f(x) = \sqrt{x_0}$$

6.2.1 Caracterization of Limits by Sequences

Theorem 6.2.1 (Caracterization of Limits by Sequences). Let $f : E \to F$ be a function defined on a neighborhood of a point $x_0 \in E$. Then, the limit of $f(x)$ as x approaches x_0 is equal to $L \in F$, i.e.:

$$\lim_{x \to x_0} f(x) = L$$

if and only if for every sequence (x_n) in E such that $\lim_{n\to+\infty} x_n = x_0$ and $x_n \neq x_0$ for all n, we have:

$$\lim_{n \to +\infty} f(x_n) = L$$

Proof. ($\Rightarrow$) Assume that $\lim_{x\to x_0} f(x) = L$. Let (x_n) be a sequence in E such that $\lim_{n\to+\infty} x_n = x_0$ and $x_n \neq x_0$ for all n. For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in E$ with $0 < |x - x_0| < \delta$, we have:

$$|f(x) - L| < \epsilon$$

Since $\lim_{n\to+\infty} x_n = x_0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have:

$$|x_n - x_0| < \delta$$

Therefore, for all $n \geq N$, we have:

$$|f(x_n) - L| < \epsilon$$

This shows that $\lim_{n\to+\infty} f(x_n) = L$.
($\Leftarrow$) By contraposition, assume that $\lim_{x\to x_0} f(x) \neq L$. Then, there exists $\epsilon_0 > 0$ such that for every $\delta > 0$, there exists $x \in E$ with $0 < |x - x_0| < \delta$ such that:

$$|f(x) - L| \geq \epsilon_0$$

For each $n \in \mathbb{N}$, let $\delta = \frac{1}{n}$. Then, there exists $x_n \in E$ with $0 < |x_n - x_0| < \frac{1}{n}$ such that:

$$|f(x_n) - L| \geq \epsilon_0$$

This defines a sequence (x_n) in E such that $\lim_{n\to+\infty} x_n = x_0$ and $x_n \neq x_0$ for all n. However, we have:

$$|f(x_n) - L| \geq \epsilon_0$$

for all n, which implies that $\lim_{n\to+\infty} f(x_n) \neq L$. This completes the proof. $\square$

Remark that the property needs to be true for every sequence converging to x_0 and not just for some particular sequences.

Example. Let f be defined by:

$$f(x) = \begin{cases} 1 & x = \frac{1}{n}, n \in \mathbb{N}^* \\ 0 & \text{otherwise} \end{cases}$$

We clearly see that $\lim_{x\to 0} f(x)$ does not exist but for the sequence $x_n = \frac{1}{n}$, we have:

$$\lim_{n\to+\infty} f(x_n) = 1$$

Showing the importance of the "for every sequence" condition in the theorem.

Let $f : E \to F$ defined on a neighborhood of x_0. Let's suppose that for all sequences (x_n) in E such that $\lim_{n\to+\infty} x_n = x_0$ and $x_n \neq x_0$ for all n, we have:

$$(f(x_n)) \text{ is convergent } \implies \lim_{x\to x_0} f(x) \text{ exists.}$$

Proof. Let's prove this by contradiction. Assume that $\lim_{x\to x_0} f(x)$ does not exist. Then, there exist two sequences (x_n) and (y_n) in E such that $\lim_{n\to+\infty} x_n = x_0$, $\lim_{n\to+\infty} y_n = x_0$, and:

$$\lim_{n\to+\infty} f(x_n) = L_1 \quad \text{and} \quad \lim_{n\to+\infty} f(y_n) = L_2$$

with $L_1 \neq L_2$. Now, we can construct a new sequence (z_n) by interleaving the terms of (x_n) and (y_n):

$$z_{2n} = x_n \quad \text{and} \quad z_{2n+1} = y_n$$

for all $n \in \mathbb{N}$. This new sequence (z_n) also converges to x_0 since both (x_n) and (y_n) converge to x_0. However, the sequence $(f(z_n))$ does not converge because:

$$\lim_{n\to+\infty} f(z_{2n}) = L_1 \quad \text{and} \quad \lim_{n\to+\infty} f(z_{2n+1}) = L_2$$

with $L_1 \neq L_2$. This contradicts our assumption that for all sequences converging to x_0, the sequence $(f(x_n))$ is convergent. Therefore, our initial assumption that $\lim_{x\to x_0} f(x)$ does not exist must be false. Hence, we conclude that $\lim_{x\to x_0} f(x)$ exists. □

Example. Let $f(x) = x^p, p \in \mathbb{N}^*$. Then, we have:

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} x^p = x_0^p$$

To show this using the caracterization by sequences, let (x_n) be an arbitrary sequence such that $\lim_{n\to+\infty} x_n = x_0$ and $x_n \neq x_0$ for all n. Additionally, let $a_n = x_n - x_0$ for all $n \in \mathbb{N}$ such that $\lim_{n\to+\infty} a_n = 0$. We have:

$$f(x_n) = (x_0 + a_n)^p = \sum_{k=0}^{p} \binom{p}{k} x_0^{p-k} a_n^k$$

Since $\lim_{n\to+\infty} a_n = 0$, we have:

$$\lim_{n\to+\infty} f(x_n) = \sum_{k=0}^{p} \binom{p}{k} x_0^{p-k} \cdot 0^k = x_0^p$$

Therefore, by the caracterization of limits by sequences, we conclude that:

$$\lim_{x\to x_0} f(x) = x_0^p$$

Definition 6.2.3 (Uniqueness of Limits).
Let $f : E \to F$ be a function defined on a neighborhood of a point $x_0 \in E$. If the limit of $f(x)$ as x approaches x_0 exists, then it is unique. In other words, if:

$$\lim_{x\to x_0} f(x) = L_1 \quad \text{and} \quad \lim_{x\to x_0} f(x) = L_2$$

then $L_1 = L_2$.

Proof. Assume that $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} f(x) = L_2$. Let (x_n) be an arbitrary sequence such that $\lim_{n\to+\infty} x_n = x_0$ and $x_n \neq x_0$ for all n. By the caracterization of limits by sequences, we have:

$$\lim_{n\to+\infty} f(x_n) = L_1 \quad \text{and} \quad \lim_{n\to+\infty} f(x_n) = L_2$$

Since the limit of a sequence is unique, we conclude that $L_1 = L_2$. Therefore, the limit of $f(x)$ as x approaches x_0 is unique. □

Theorem 6.2.2 (Cauchy Criterion for Limits of Functions). Let $f : E \to F$ be a function defined on a neighborhood of a point $x_0 \in E$. Then, the limit of $f(x)$ as x approaches x_0 exists if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in E$ with $0 < |x - x_0| < \delta$ and $0 < |y - x_0| < \delta$, we have:

$$|f(x) - f(y)| < \epsilon$$

6.2.2 Operations on Limits

Definition 6.2.4. Let $f : E \to F$ and $g : E \to F$ be two functions defined on a neighborhood of a point $x_0 \in E$ such that the limits $\lim_{x\to x_0} f(x) = l_1$ and $\lim_{x\to x_0} g(x) = l_2$, then we can define the following operations on their limits:

- Sum: $\lim_{x\to x_0}(\alpha f(x) + \beta g(x)) = \alpha l_1 + \beta l_2$
- Product: $\lim_{x\to x_0}(f(x) \cdot g(x)) = l_1 \cdot l_2$
- Quotient: If $l_2 \neq 0$, then $\lim_{x\to x_0}\left(\frac{f(x)}{g(x)}\right) = \frac{l_1}{l_2}$

For all polynomial function P and rational function $R = \frac{P_1}{P_2}$ (with $P_2 \neq 0$) defined for all $x_0 \in \mathbb{R}$ except the roots of P_2, we have:

$$\lim_{x\to x_0} P(x) = P(x_0) \quad \text{and} \quad \lim_{x\to x_0} R(x) = R(x_0)$$

Example. Let's compute the limit:

$$\lim_{x\to 1} \frac{2x^5 - 3x^3 + 2x + 1}{x^4 + 2x^2 - 6x}$$

We have:

$$\lim_{x\to 1}(2x^5 - 3x^3 + 2x + 1) = 2(1)^5 - 3(1)^3 + 2(1) + 1 = 2 - 3 + 2 + 1 = 2$$

and:

$$\lim_{x\to 1}(x^4 + 2x^2 - 6x) = (1)^4 + 2(1)^2 - 6(1) = 1 + 2 - 6 = -3$$

Therefore, we can apply the quotient rule:

$$\lim_{x\to 1} \frac{2x^5 - 3x^3 + 2x + 1}{x^4 + 2x^2 - 6x} = \frac{2}{-3} = -\frac{2}{3}$$

Theorem 6.2.3 (Sandwich Theorem for Functions). Let $f, g, h : E \to F$ be three functions defined on a neighborhood of a point $x_0 \in E$. If there exists $\alpha > 0$ such that for all $x \in \{x \in E : 0 < |x - x_0| \leq \alpha\}$, we have:

$$f(x) \leq g(x) \leq h(x)$$

and if:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L$$

then:

$$\lim_{x \to x_0} g(x) = L$$

Proof. Let $(a_n) \in \{x \in E - \{x_0\}\}$ be a sequence such that $\lim_{n \to +\infty} a_n = x_0$. There exists $m \in \mathbb{N} : \forall n \geq m$ we have:

$$0 < |a_n - x_0| < \alpha$$

Thus, for all $n \geq m$, we have:

$$f(a_n) \leq g(a_n) \leq h(a_n)$$

Since $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} h(x) = L$, by the caracterization of limits by sequences, we have:

$$\lim_{n \to +\infty} f(a_n) = L \quad \text{and} \quad \lim_{n \to +\infty} h(a_n) = L$$

Therefore, by the Sandwich Theorem for Sequences, we conclude that:

$$\lim_{n \to +\infty} g(a_n) = L$$

Since the sequence (a_n) was arbitrary, by the caracterization of limits by sequences, we conclude that:

$$\lim_{x \to x_0} g(x) = L$$

□

Example. Let $g : \mathbb{R} \to \mathbb{R}$ a function such that $\lim_{x \to 0} g(x) = 0$, then:

$$\lim_{x \to 0} \sqrt{1 + g(x)} = 1$$

To show this, we can use the Sandwich Theorem. We know that:

$$1 - |t| \leq \sqrt{1 + t} \leq \left|1 + \frac{1}{2}t\right|, \forall t \geq -1$$

Since:

$$\sqrt{1 + t} \leq \sqrt{1 + t + \frac{t^2}{4}} = \sqrt{\left(1 + \frac{t}{2}\right)^2} = \left|1 + \frac{1}{2}t\right|, \forall t \in \mathbb{R}$$

and:

$$t \geq 0 \implies 1 - |t| = 1 - t \leq 1 \leq \sqrt{1 + t} \quad \text{and} \quad -1 < t < 0 \implies 1 - |t| = 1 + t \leq \sqrt{1 + t}$$

Thus, since $\lim_{x\to 0} g(x) = 0$, there exists $\delta > 0$ such that if $|x| < \delta$, we have:

$$|g(x)| \leq 1 \quad \text{and} \quad g(x) \geq -1$$

Therefore, we have:

$$\underbrace{1 - |g(x)|}_{\to 1\ (x\to 0)} \leq \sqrt{1+g(x)} \leq \underbrace{\left|1 + \frac{1}{2}g(x)\right|}_{\to 1\ (x\to 0)}$$

By the Sandwich Theorem, we conclude that:

$$\lim_{x\to 0} \sqrt{1+g(x)} = 1$$

Example. Let $f(x) = x^2 \cdot \cos\left(\frac{1}{x}\right)$ be a function not defined in $x = 0$. We want to compute:

$$\lim_{x\to 0} f(x)$$

We have:

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \quad \underbrace{\Longrightarrow}_{x^2 \geq 0} \quad -x^2 \leq x^2 \cdot \cos\left(\frac{1}{x}\right) \leq x^2$$

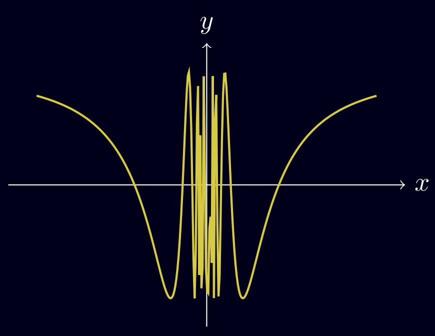
Since:

$$\lim_{x\to 0} -x^2 = 0 \quad \text{and} \quad \lim_{x\to 0} x^2 = 0$$

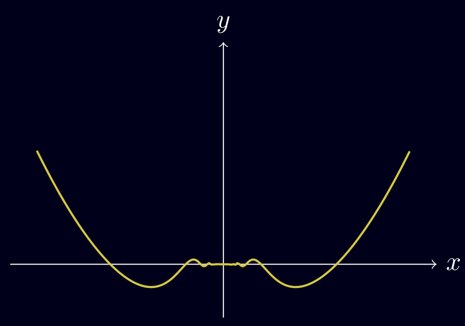
By the Sandwich Theorem, we conclude that:

$$\lim_{x\to 0} x^2 \cdot \cos\left(\frac{1}{x}\right) = 0$$

From the example above, we can also visualize the functions:



Graph of $\cos\left(\frac{1}{x}\right)$



Graph of $x^2 \cdot \cos\left(\frac{1}{x}\right)$

Example. Let's compute the limit:

$$\lim_{x\to 0} \frac{\sin x}{x}$$

For x close to 0 ($x \neq 0$ thus $\sin x \neq 0$), we can bound $\sin x$ by:

$$|\sin x| \leq |x| \leq |\tan x|$$

Since $\sin x \neq 0$, we can divide the inequalities by $|\sin x|$:

$$1 \leq \left|\frac{x}{\sin x}\right| \leq \left|\frac{1}{\cos x}\right| \quad \iff \quad 1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

We have:

$$\cos 2\alpha = (\cos\alpha)^2 - (\sin\alpha)^2 = 1 - 2(\sin\alpha)^2 \quad \implies \quad 1 - \cos 2\alpha = 2(\sin\alpha)^2$$

and:

$$|1-\cos x| = 2\left|\left(\sin\frac{x}{2}\right)^2\right| = \left(\sin\frac{x}{2}\right)^2 \leq 2\left|\frac{x}{2}\right|^2 = \frac{x^2}{2} \quad \implies \quad |\cos x - 1| \leq \frac{x^2}{2}$$

By the Squeeze Theorem, we have:

$$0 \leq |1-\cos x| \leq \frac{x^2}{2} \quad \implies \quad \lim_{x\to 0} |1-\cos x| = 0 \quad \implies \quad \lim_{x\to 0} \cos x = 1$$

and:

$$\lim_{x\to 0} \frac{1}{\cos x} = 1$$

Finally, by the Squeeze Theorem, we conclude that:

$$1 \leq \frac{x}{\sin x} \leq \underbrace{\frac{1}{\cos x}}_{=1} \quad \implies \quad \lim_{x\to 0} \frac{x}{\sin x} = 1 \quad \implies \quad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

6.2.3 Limits of Composite Functions

Theorem 6.2.4 (Limits of Composite Functions). Let $f : E \to F$ and $g : F \to G$ be two functions such that $\lim_{x\to x_0} f(x) = y_0$ and $\lim_{y\to y_0} g(y) = L$. If there exists $\alpha > 0$ such that for all $x \in \{y \in E : 0 < |y - x_0| < \alpha\}$, we have $f(x) \neq y_0$, then:

$$\lim_{x\to x_0} g(f(x)) = L$$

Remark that the condition $f(x) \neq y_0$ is necessary because $g(y_0)$ might not be defined and thus we cannot have $f(x)$ approaching y_0 directly.

Proof. Let $\lim_{y\to y_0} g(y) = L$, we have for a given $\epsilon > 0$, there exists $\delta_1 > 0$ such that for all $y \in F$ with $0 < |y - y_0| < \delta_1$, we have:

$$|g(y) - L| < \epsilon$$

Since $\lim_{x\to x_0} f(x) = y_0$, there exists $\delta_2 > 0$ such that for all $x \in E$ with $0 < |x - x_0| < \delta_2$, we have:

$$|f(x) - y_0| < \delta_1$$

Let $\delta = \min(\alpha, \delta_2)$, then for all $x \in E$ with $0 < |x - x_0| < \delta$, we have:

$$|f(x) - y_0| < \delta_1$$

and since $f(x) \neq y_0$, we can apply the first inequality:

$$|g(f(x)) - L| < \epsilon$$

Therefore, we conclude that:

$$\lim_{x \to x_0} g(f(x)) = L$$

$\square$

This theorem allows to change the variable in a limit computation.

Example. Let's compute the limit:

$$\lim_{x \to 0} \frac{\cos(4x) - \cos(2x)}{x^2}$$

If we compute the limit directly, we get the indeterminate form $\frac{0}{0}$. To resolve this, we can use the trigonometric identity $(\cos(2\alpha) = 1 - 2(\sin\alpha)^2)$, we get:

$$\lim_{x \to 0} \frac{1 - (\sin(2x))^2 - 1 + 2(\sin x)^2}{x^2} = \lim_{x \to 0} 2 \cdot \frac{(\sin x)^2 - (\sin(2x))^2}{x^2}$$

Thus, we can rewrite the limit as:

$$\lim_{x \to 0} 2 \left(\underbrace{\left(\frac{\sin x}{x}\right)^2}_{\to 1^2} - \underbrace{\left(\frac{\sin(2x)}{2x}\right)^2}_{\to 1^2} \cdot 4 \right) = 2(1 - 4) = -6$$

Example. Let's compute the limit:

$$\lim_{x \to 2} \frac{\sqrt{x+2} - \sqrt{2x}}{\sqrt{x-1} - 1}$$

Again, if we compute the limit directly, we get the indeterminate form $\frac{0}{0}$. To resolve this, we can multiply the numerator and denominator by the conjugate of each:

$$\lim_{x \to 2} \frac{(\sqrt{x+2} - \sqrt{2x})(\sqrt{x+2} + \sqrt{2x})}{(\sqrt{x-1} - 1)(\sqrt{x-1} + 1)} = \lim_{x \to 2} \frac{(x+2) - 2x}{(x-1) - 1} \cdot \frac{\sqrt{x-1} + 1}{\sqrt{x+2} + \sqrt{2x}}$$

Thus, we can rewrite the limit as:

$$\lim_{x \to 2} \frac{2 - x}{x - 2} \cdot \lim_{x \to 2} \frac{\sqrt{x-1} + 1}{\sqrt{x+2} + \sqrt{2x}} = -1 \cdot \frac{\sqrt{2-1} + 1}{\sqrt{2+2} + \sqrt{4}} = -\frac{1+1}{2+2} = -\frac{1}{2}$$

Remark that by the theorem above, $\lim_{x \to 0} \frac{\sin(t(x))}{t(x)} = 1$ if $\lim_{x \to 0} t(x) = 0$ and $t(x) \neq 0$ for $x \neq 0$ close to 0.

Example. Let's compute the limit:

$$\lim_{x\to 3}\frac{\sin(\sqrt{x+1}-2)}{x-3}$$

Since $\lim_{x\to 3}\sqrt{x+1}-2=0$, we can rewrite the limit as:

$$\lim_{x\to 3}\underbrace{\frac{\sin(\sqrt{x+1}-2)}{\sqrt{x+1}-2}}_{\to 1}\cdot\frac{\sqrt{x+1}-2}{x-3}=\lim_{x\to 3}\frac{\sin(\sqrt{x+1}-2)}{\sqrt{x+1}-2}\cdot\frac{x+1-4}{\sqrt{x+1}+2}\cdot\frac{1}{x-3}$$

Thus, we can rewrite the limit as:

$$\lim_{x\to 3}\frac{\sin(\sqrt{x+1}-2)}{\sqrt{x+1}-2}\cdot\frac{x-3}{x-3}\cdot\frac{1}{\sqrt{x+1}+2}=1\cdot 1\cdot\frac{1}{\sqrt{4}+2}=\frac{1}{4}$$

Remark that the sin simplification would not be valid, for example, if $t(x)=x^2\cos(\frac{1}{x})$ because the function $t(x)$ is not defined in the neighborhood of 0.

Example. Let's prove that the limit $\lim_{x\to 0}\sin(\frac{1}{x})$ does not exists. To show this, we can consider two sequences (a_n) and (b_n) defined by:

$$a_n=\frac{1}{\pi n}\quad\text{and}\quad b_n=\frac{1}{\frac{\pi}{2}+2\pi n}$$

for all $n\in\mathbb{N}^*$. We have:

$$\lim_{n\to+\infty}a_n=\lim_{n\to+\infty}b_n=0$$

However, we have:

$$\lim_{n\to+\infty}\sin\left(\frac{1}{a_n}\right)=\lim_{n\to+\infty}\sin(\pi n)=0$$

and:

$$\lim_{n\to+\infty}\sin\left(\frac{1}{b_n}\right)=\lim_{n\to+\infty}\sin\left(\frac{\pi}{2}+2\pi n\right)=1$$

Since the two sequences $(\sin(\frac{1}{a_n}))$ and $(\sin(\frac{1}{b_n}))$ converge to different limits, by the caracterization of limits by sequences, we conclude that the limit $\lim_{x\to 0}\sin(\frac{1}{x})$ does not exists.

In general, to prove that a limit does not exits, two sequences converging to the same point can be used to show that the function values converge to different limits.

Example. Let's compute the limit:

$$\lim_{x\to 0}x\cdot\sin\left(\frac{1}{x}\right)\quad\text{where}\quad D(f)=\mathbb{R}-\{0\}$$

By using the Squeeze Theorem, we have:

$$-1 \leq \sin\frac{1}{x} \leq 1 \quad \Longleftrightarrow \quad 0 \leq \left|\sin\frac{1}{x}\right| \leq 1$$

Multplying by $|x|$ (which is positive when $x \to 0$), we get:

$$0 \leq \left|x \cdot \sin\frac{1}{x}\right| \leq \underbrace{|x|}_{\to 0}$$

Thus, by the Squeeze Theorem, we conclude that:

$$\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0$$

6.2.4 Limits at Infinity and Infinite Limits

Definition 6.2.5 (Neighborhoods of Infinity).
Let $f : E \to F$ is defined on a neighborhood of $+\infty$ (resp. $-\infty$) if there exists $\alpha \in \mathbb{R}$: $(\alpha, +\infty) \subset E$ (resp. $(-\infty, \alpha) \subset E$).

Definition 6.2.6 (Limit at Infinity).
Let $f : E \to F$ be a function defined on a neighborhood of $+\infty$ (resp. $-\infty$). It is said that the limit of $f(x)$ as x approaches $+\infty$ (resp. $-\infty$) is equal to $L \in F$ if for every $\epsilon > 0$, there exists $M > 0$ such that for all $x \in E$ with $x > M$ (resp. $x < -M$), we have:

$$|f(x) - L| < \epsilon$$

This is denoted by:

$$\lim_{x \to +\infty} f(x) = L \quad \left(\text{resp. } \lim_{x \to -\infty} f(x) = L\right)$$

In this case, it is said that the function f has a horizontal asymptote $y = L$ at $+\infty$ (resp. $-\infty$).

Example. Let's prove that the limit:

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

To show this, let $\epsilon > 0$ be given. We need to find $M > 0$ such that for all $x > M$, we have:

$$\left|\frac{1}{x^2} - 0\right| < \epsilon$$

We have:
$$\left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2} < \epsilon \quad \iff \quad x^2 > \frac{1}{\epsilon} \quad \iff \quad x > \frac{1}{\sqrt{\epsilon}}$$
Thus, we can choose $M = \frac{1}{\sqrt{\epsilon}}$. Therefore, for all $x > M$, we have:
$$\left|\frac{1}{x^2} - 0\right| < \epsilon$$
Thus, we conclude that:
$$\lim_{x\to\infty} \frac{1}{x^2} = 0$$

Definition 6.2.7 (Infinite Limits).
Let $f : E \to F$ be a function defined on a neighborhood of $x_0 \in E$. It is said that the limit of $f(x)$ as x approaches x_0 is equal to $+\infty$ (resp. $-\infty$) if for every $M > 0$, there exists $\delta > 0$ such that for all $x \in E$ with $0 < |x - x_0| < \delta$, we have:
$$f(x) > M \quad (\text{resp. } f(x) < -M)$$
This is denoted by:
$$\lim_{x\to x_0} f(x) = +\infty \quad (\text{resp. } \lim_{x\to x_0} f(x) = -\infty)$$
In this case, it is said that the function f has a vertical asymptote $x = x_0$.

Example. Let's prove the following limit:
$$\lim_{x\to 0} \frac{1}{x^2} = +\infty$$
To show this, let $M > 0$ be given. We need to find $\delta > 0$ such that for all $x \in E$ with $0 < |x - 0| < \delta$, we have:
$$\frac{1}{x^2} > M$$
We have:
$$\frac{1}{x^2} > M \quad \iff \quad |x| < \frac{1}{\sqrt{M}}$$
Thus, we can choose $\delta = \frac{1}{\sqrt{M}}$. Therefore, for all $x \in E$ with $0 < |x| < \delta$, we have:
$$\frac{1}{x^2} > M$$
Thus, we conclude that:
$$\lim_{x\to 0} \frac{1}{x^2} = +\infty$$

Example. Let's show that the following limit does not exist:

$$\lim_{x \to 0} \frac{1}{x}$$

To show this, we can consider two sequences (a_n) and (b_n) defined by:

$$a_n = \frac{1}{n} \quad \text{and} \quad b_n = -\frac{1}{n}$$

for all $n \in \mathbb{N}^*$. We have:

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} b_n = 0$$

However, we have:

$$\lim_{n \to +\infty} \frac{1}{a_n} = \lim_{n \to +\infty} n = +\infty$$

and:

$$\lim_{n \to +\infty} \frac{1}{b_n} = \lim_{n \to +\infty} -n = -\infty$$

Since the two sequences $(\frac{1}{a_n})$ and $(\frac{1}{b_n})$ converge to different limits, by the caracterization of limits by sequences, we conclude that the limit $\lim_{x \to 0} \frac{1}{x}$ does not exist.

Definition 6.2.8 (Infite Limits at Infinity).
Let $f : E \to F$ be a function defined on a neighborhood of $+\infty$ (resp. $-\infty$). It is said that the limit of $f(x)$ as x approaches $+\infty$ (resp. $-\infty$) is equal to $+\infty$ (resp. $-\infty$) if for every $M > 0$, there exists $X > 0$ such that for all $x \in E$ with $x > X$ (resp. $x < -X$), we have:

$$f(x) > M \quad (\text{resp. } f(x) < -M)$$

This is denoted by:

$$\lim_{x \to +\infty} f(x) = +\infty \quad (\text{resp. } \lim_{x \to -\infty} f(x) = -\infty)$$

Note that all results shown for limits at finite points can be adapted to limits at infinity and vice versa.

Example. Some examples of infinite limits at infinity are:

$$\lim_{x \to +\infty} x = +\infty, \quad \lim_{x \to -\infty} x = -\infty, \quad \lim_{x \to +\infty} e^x = +\infty, \quad \lim_{x \to -\infty} e^x = 0$$

6.2.5 Indeterminate Forms

When computing limits, we can encounter some indeterminate forms such as:

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $0 \cdot \infty$
- $\infty - \infty$

- 0^0 (will be seen later)
- 1^∞ (will be seen later)
- ∞^0 (will be seen later)

Example. Let's compute the limit:

$$\lim_{x\to\infty}(\sqrt{x^2+2x}-x)$$

If we compute the limit directly, we get the indeterminate form $\infty-\infty$. To resolve this, we can multiply the expression by its conjugate:

$$\lim_{x\to\infty}(\sqrt{x^2+2x}-x)\cdot\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}=\lim_{x\to\infty}\frac{(x^2+2x)-x^2}{\sqrt{x^2+2x}+x}=\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+2x}+x}$$

Thus, we can rewrite the limit as:

$$\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{2}{x}}+1}=\frac{2}{\sqrt{1+0}+1}=\frac{2}{2}=1$$

Example. Let's compute the limit:

$$\lim_{x\to-\infty}(\sqrt{x^2+2x}+x)$$

If we compute the limit directly, we get the indeterminate form $\infty-\infty$. To resolve this, we can multiply the expression by its conjugate:

$$\lim_{x\to-\infty}(\sqrt{x^2+2x}+x)\cdot\frac{\sqrt{x^2+2x}-x}{\sqrt{x^2+2x}-x}=\lim_{x\to-\infty}\frac{(x^2+2x)-x^2}{\sqrt{x^2+2x}-x}=\lim_{x\to-\infty}\frac{2x}{\sqrt{x^2+2x}-x}$$

Thus, we can rewrite the limit as:

$$\lim_{x\to-\infty}\frac{2x}{\underbrace{\sqrt{x^2}}_{|x|=-x\ (x\to-\infty)}\sqrt{1+\frac{1}{2}}-x}==\lim_{x\to-\infty}\frac{2x}{-x\left(\sqrt{1+\frac{2}{x}}+1\right)}=-1$$

Example. Let's compute the limit:

$$\lim_{x\to1}\frac{\sin\left(\frac{\pi x}{2}\right)+x^2-2x}{(x-1)^2}$$

If we compute the limit directly, we get the indeterminate form $\frac{0}{0}$. To resolve this, we can make a change of variable $t=x-1$:

$$\lim_{t\to0}\frac{\sin\left(\frac{\pi(t+1)}{2}\right)+(t+1)^2-2(t+1)}{t^2}=\lim_{t\to0}\frac{\sin\left(\frac{\pi t}{2}+\frac{\pi}{2}\right)+t^2+2t+1-2t-2}{t^2}$$

By using the trigonometric identity $\sin(\alpha + \frac{\pi}{2}) = \cos\alpha$, we get:

$$= \lim_{t\to 0} \frac{\cos\left(\frac{\pi t}{2}\right) + t^2 - 1}{t^2}$$

Again, by using the trigonometric identity $\cos\alpha = 1 - 2(\sin\frac{\alpha}{2})^2$, we get:

$$= \lim_{t\to 0} \frac{-2\left(\sin\frac{\pi t}{4}\right)^2 + t^2}{t^2} = \lim_{t\to 0}\left(1 - 2\underbrace{\frac{\left(\sin(\frac{\pi t}{4})\right)^2}{\left(\frac{\pi t}{4}\right)^2}}_{\to 1}\frac{\left(\frac{\pi t}{4}\right)^2}{t^2}\right) = 1 - \frac{\pi^2}{8}$$

6.2.6 Properties of Infinite Limits

The properties of limits shown previously can be adapted to infinite limits. Here are some examples:

- If $\lim_{x\to x_0} f(x) = \pm\infty$ and $\lim_{x\to x_0} g(x) = \pm\infty$, then $\lim_{x\to x_0}(f(x) + g(x)) = \pm\infty$
- If $\lim_{x\to x_0} f(x) = +\infty$ and $\lim_{x\to x_0} g(x) = L \in \mathbb{R}^*$, then $\lim_{x\to x_0}(f(x)\cdot g(x)) = +\infty$ if $L > 0$ and $-\infty$ if $L < 0$
- If $\lim_{x\to x_0} f(x) = \pm\infty$ and $g(x)$ is bounded in a neighborhood of x_0, then $\lim_{x\to x_0}(f(x) \pm g(x)) = \pm\infty$
- If $\lim_{x\to x_0} f(x) = \pm\infty$, then $\lim_{x\to x_0} \frac{1}{f(x)} = 0$
- If $\lim_{x\to x_0} f(x) = 0$ and $f(x) \neq 0$ in a neighborhood of x_0, then $\lim_{x\to x_0} \frac{1}{f(x)} = +\infty$ if $f(x) > 0$ in a neighborhood of x_0, $-\infty$ if $f(x) < 0$ in a neighborhood of x_0 and does not exist otherwise
- If $\lim_{x\to x_0} = +\infty$ $(-\infty)$ and $g(x) \geq f(x)$ $(g(x) \leq f(x))$ for all x in a neighborhood of x_0, then $\lim_{x\to x_0} g(x) = +\infty$ $(-\infty)$ (Squeeze Theorem)

Note that these properties are aslo valid for limits at infinity.

6.2.7 Right and Left Limits

Definition 6.2.9 (Right and Left Limits).
Let $f : E \to F$ be a function and $x_0 \in \mathbb{R}$ be an accumulation point of E.
It is said that the **right limit** of $f(x)$ as x approaches x_0 is equal to $L \in F$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in E$ with $x_0 < x < x_0 + \delta$, we have:

$$|f(x) - L| < \epsilon$$

This is denoted by:

$$\lim_{x\to x_0^+} f(x) = L$$

It is said that the **left limit** of $f(x)$ as x approaches x_0 is equal to $L \in F$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in E$ with $x_0 - \delta < x < x_0$, we have:

$$|f(x) - L| < \epsilon$$

This is denoted by:

$$\lim_{x \to x_0^-} f(x) = L$$

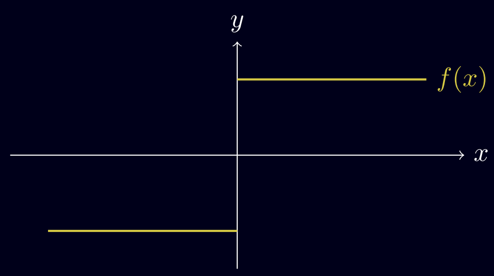
Remark that a limit $\lim_{x \to x_0} f(x) = L$ exists if and only if both right and left limits exist and are equal:

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$$

Example. Let's consider the function:

$$f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

Graphically, this function looks like:



We have:

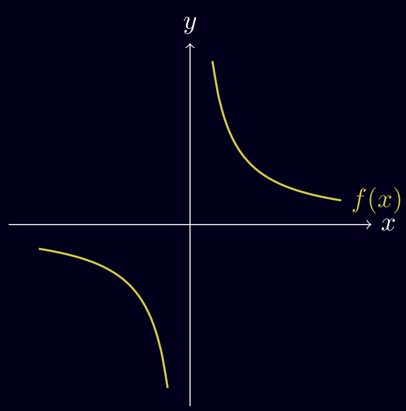
$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1$$

Thus, the limit $\lim_{x \to 0} f(x)$ does not exist.

Example. Let consider the function $f(x) = \frac{1}{x}$. We have:

$$\lim_{x \to 0^+} f(x) = +\infty \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -\infty$$

Graphically, this function looks like:



Thus, the limit $\lim_{x\to 0} f(x)$ does not exist.

6.3 Exponential and Logarithmic Functions

Definition 6.3.1 (e).
The number e is defined as the unique positive number such that:

$$\sum_{n=0}^{+\infty} \frac{x^n}{n!} = e^x$$

Which converges for all $x \in \mathbb{R}$ (D'Alembert's ratio test).

By convention, we define $0! = 1$ and $0^0 = 1$. The following properties hold:

- $e^{x+y} = e^x \cdot e^y$ for all $x, y \in \mathbb{R}$
- $e^{-x} = \frac{1}{e^x}$ for all $x \in \mathbb{R}$
- $e^x > 0$ for all $x \in \mathbb{R}$

Proof. **1. Property**:

$$e^x \cdot e^y = \left(\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right) \cdot \left(\sum_{m=0}^{+\infty} \frac{y^m}{m!}\right) = \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} \frac{x^n y^m}{n!m!}$$

By rearranging the terms, we get:

$$= \sum_{k=0}^{+\infty}\sum_{n=0}^{k} \frac{x^n y^{k-n}}{n!(k-n)!} = \sum_{k=0}^{+\infty} \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} x^n y^{k-n}$$

By using the binomial theorem, we have:

$$= \sum_{k=0}^{+\infty} \frac{(x+y)^k}{k!} = e^{x+y}$$

2. Property:

$$e^x \cdot e^{-x} = e^{x-x} = e^0 = 1 \quad \Longrightarrow \quad e^{-x} = \frac{1}{e^x}$$

3. Property: Since all terms of the series defining e^x are positive, we have:

$$e^x = e^{\frac{x}{2}} \cdot e^{\frac{x}{2}} = \left(e^{\frac{x}{2}}\right)^2 > 0$$

□

6.3.1 Properties of Exponential Function

These properties can be proved using the definition of e^x:

- $\lim_{x\to\infty} e^x = \lim_{x\to\infty} \sum_{k=0}^{\infty} \frac{x^k}{k!} \geq \lim_{x\to\infty}(1+x) = +\infty$
- $\lim_{x\to-\infty} e^x = \lim_{y\to\infty} e^{-y} = \lim_{y\to\infty} \frac{1}{e^y} = 0$
- e^x is strictly increasing on $\mathbb{R}$. Let $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$. We have:

$$e^{x_2} - e^{x_1} = e^{x_1}(e^{x_2 - x_1} - 1) > 0$$

Example. Let's prove the following limit:

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

To prove this, we can use the definition of e^x:

$$\begin{aligned}\left|\frac{e^x-1}{x}-1\right| &= \left|\frac{\sum_{k=0}^{\infty}\frac{x^k}{k!}-1}{x}-1\right| = \left|\frac{\sum_{k=1}^{\infty}\frac{x^k}{k!}}{x}-1\right| = \left|\sum_{k=1}^{\infty}\frac{x^{k-1}}{k!}-1\right| \\ &= \left|\sum_{k=2}^{\infty}\frac{x^{k-1}}{k!}\right| \\ &\le \sum_{k=2}^{\infty}\frac{|x|^{k-1}}{k!} = |x|\sum_{k=2}^{\infty}\frac{|x|^{k-2}}{k!} \\ &\le |x|\sum_{k=2}^{\infty}\frac{|x|^{k-2}}{(k-2)!} = |x|\sum_{m=0}^{\infty}\frac{|x|^m}{m!} = |x|e^{|x|}\end{aligned}$$

By the Squeeze Theorem, we conclude that:

$$0<\left|\frac{e^x-1}{x}-1\right|<\underbrace{|x|e^{|x|}}_{\to 0}\implies \lim_{x\to 0}\frac{e^x-1}{x}=1$$

Remark that this limit $\lim_{x\to a}\frac{e^{g(x)}-1}{g(x)}$ also converges to 1 for any function $g(x)$ such that $\lim_{x\to a}g(x)=0$ and $g(x)\neq 0$ in the neighborhood of $x=a$.

6.3.2 Logarithmic Function

Definition 6.3.2 (Natural Logarithm).
The natural logarithm function $\ln:(0,+\infty)\to\mathbb{R}$ is defined as the inverse function of the exponential function $e^x:\mathbb{R}\to(0,+\infty)$.

6.3.3 Properties of Logarithmic Function

These properties can be proved using the definition of $\ln x$:

- $e^{\ln x}=x$ for all $x>0$, $\ln(e^x)=x$ for all $x\in\mathbb{R}$
- $\ln(xy)=\ln x+\ln y$ for all $x,y>0$
- $\ln\left(\frac{x}{y}\right)=\ln x-\ln y$ for all $x,y>0$
- $\ln(x^r)=r\ln x$ for all $r\in\mathbb{R}$
- $\ln(1)=0$, $\ln(e)=1$

6.4 List of Important Limits

Here is a list of some important limits related to exponential and logarithmic functions:

- $\lim_{x\to 0} \frac{e^x-1}{x} = 1$
- $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$
- $\lim_{x\to a} \frac{e^{t(x)}-1}{t(x)} = 1$ if $\lim_{x\to a} t(x) = 0$ and $t(x) \neq 0$ in a neighborhood of $x = a$
- $\lim_{x\to a} \frac{\ln(1+t(x))}{t(x)} = 1$ if $\lim_{x\to a} t(x) = 0$ and $t(x) \neq 0$ in a neighborhood of $x = a$
- $\lim_{x\to a} (1+t(x))^{\frac{1}{t(x)}} = e$ if $\lim_{x\to a} t(x) = 0$ and $t(x) \neq 0$ in a neighborhood of $x = a$

6.5 Continuity

Definition 6.5.1 (Continuity at a Point).
Let $f : E \to F$ be a function and $x_0 \in E$. The function f is said to be continuous at point x_0 if:

$$\lim_{x\to x_0} f(x) = f(x_0)$$

Remark that this condition could be spit into three parts:

- $f(x_0)$ is defined (i.e., $x_0 \in E$)
- $\lim_{x\to x_0} f(x)$ exists
- $\lim_{x\to x_0} f(x) = f(x_0)$

Example. The following examples illustrate the concept of continuity at a point:

- $f(x) = x^p, p \in \mathbb{N}$ is continous at every point $x_0 \in \mathbb{R}$, since $\lim_{x\to a} x^p = a^p$ for all $a \in \mathbb{R}$.
- All polynomial functions are continuous at every point $x_0 \in \mathbb{R}$.
- Any rational function $\frac{P(x)}{Q(x)}$ is continuous at every point $x_0 \in \mathbb{R}$ such that $Q(x_0) \neq 0$.
- $f(x) = \sqrt[p]{x}$ is continuous at every point $x_0 \in (0, +\infty)$ for all $p \in \mathbb{N}^*$.
- $f(x) = \sin(x)$ and $g(x) = \cos(x)$ are continuous at every point $x_0 \in \mathbb{R}$.

Theorem 6.5.1 (Cauchy's Criterion for Continuity). Let $f : E \to F$ be a function defined in a neighborhood of $x = x_0$ and at x_0. The function f is continuous at point x_0 if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $|x - x_0| < \delta$ and $|y - x_0| < \delta$, we have:

$$|f(x) - f(y)| < \epsilon$$

Proof. ($\Rightarrow$) **Direction**: Assume that f is continuous at point x_0. Let $\epsilon > 0$ be given. By the definition of continuity, there exists $\delta > 0$ such that for all $x \in E$ with $|x - x_0| < \delta$, we have:

$$|f(x) - f(x_0)| < \frac{\epsilon}{2}$$

Now, let $x, y \in E$ such that $|x - x_0| < \delta$ and $|y - x_0| < \delta$. We have:

$$|f(x) - f(y)| = |f(x) - f(x_0) + f(x_0) - f(y)| \leq |f(x) - f(x_0)| + |f(y) - f(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, the condition is satisfied.
($\Leftarrow$) Direction: Assume that for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $|x - x_0| < \delta$ and $|y - x_0| < \delta$, we have:

$$|f(x) - f(y)| < \epsilon$$

To show that f is continuous at point x_0, let $\epsilon > 0$ be given. By assumption, there exists $\delta > 0$ such that for all $x, y \in E$ with $|x - x_0| < \delta$ and $|y - x_0| < \delta$, we have:

$$|f(x) - f(y)| < \epsilon$$

In particular, for any $x \in E$ with $|x - x_0| < \delta$, we can choose $y = x_0$. Thus, we have:

$$|f(x) - f(x_0)| < \epsilon$$

Therefore, by the definition of continuity, we conclude that f is continuous at point x_0. □

Definition 6.5.2 (Continuity on the left and on the right).
Let $f : E \to F$ be a function and $x_0 \in E$. The function f is said to be continuous on the **right** at point x_0 if:

$$\lim_{x \to x_0^+} f(x) = f(x_0)$$

The function f is said to be continuous on the **left** at point x_0 if:

$$\lim_{x \to x_0^-} f(x) = f(x_0)$$

Remark that f is continuous at point x_0 if and only if it is continuous on the left and on the right at point x_0.

Example. Let's study the continuity of a function f in $x = 0$. Consider the function:

$$f(x) = \begin{cases} 2x + 1, & x \geq 0 \\ \frac{\sin x}{x}, & x < 0 \end{cases}$$

Let's compute the right limit:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2x + 1) = 1$$

Let's compute the left limit:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin x}{x} = 1$$

Since both limits are the same, we conclude that f is continuous at point $x = 0$.

Example. Let's study the continuity of a function f in $x = 0$. Consider the function:

$$f(x) = \begin{cases} 2x+1, & x > 0 \\ 2, & x = 0 \\ \frac{\sin x}{x}, & x < 0 \end{cases}$$

Let's compute the right limit:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2x+1) = 1$$

Let's compute the left limit:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin x}{x} = 1$$

Since both limits are the same but different from $f(0) = 2$, we conclude that f is not continuous at point $x = 0$.

6.5.1 Operations on Continuous Functions

Let $f, g : E \to F$ be two functions continuous at point $x_0 \in E$. The following operations produce new functions that are also continuous at point x_0:

- The sum function $h(x) = \alpha f(x) + \beta g(x)$ is continuous at point x_0 (for any $\alpha, \beta \in \mathbb{R}$).
- The product function $h(x) = f(x) \cdot g(x)$ is continuous at point x_0.
- The quotient function $h(x) = \frac{f(x)}{g(x)}$ is continuous at point x_0 if $g(x_0) \neq 0$.
- If $f : E \to F$ and $g : G \to H$, $f(E) \subset G$, f is continuous at point $x_0 \in E$ and g is continuous at point $f(x_0) \in G$, then the composition function $h(x) = g(f(x))$ is continuous at point x_0.

Remark that if $(g \circ f)$ is continuous at point x_0, it does not necessarily imply that f is continuous at point x_0 or that g is continuous at point $f(x_0)$.

Example. Let f be a function defined as:

$$f(x) = \frac{\sqrt{e^{2x^2+\sin x}} + 8x^2 + \ln(x^2+1)}{x^4+8}$$

We can see that f is a composition of continuous functions (polynomial, exponential, square root, logarithm) and thus it is continuous for all $x \in \mathbb{R}$.

Example. Let consider the functions:

$$f(x) = \begin{cases} |x|, & x \neq 0 \\ -1, & x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0, & x \geq -1 \\ 1, & x < -1 \end{cases}$$

We have:

$$(g \circ f)(x) = g(f(x)) = 0 \quad \text{for all } x \in \mathbb{R}$$

Thus, $(g \circ f)(x)$ is continuous at point $x = 0$, but $f(x)$ is not continuous at point $x = 0$ and $g(x)$ is not continuous at point $f(0) = -1$.

6.5.2 Extension by Continuity

Definition 6.5.3 (Extension by Continuity).
Let $f : E \to F$ be a function and x_0 be an accumulation point of E but not in E. If the limit $\lim_{x \to x_0} f(x) = L$ exists, we can define a new function $\tilde{f} : E \cup \{x_0\} \to F$ as:

$$\tilde{f}(x) = \begin{cases} f(x), & x \in E \\ L, & x = x_0 \end{cases}$$

The function $\tilde{f}$ is called the extension by continuity of f at point x_0.

Example. Let's consider the function:

$$f(x) = x \sin\left(\frac{1}{x}\right)$$

defined on $E = \mathbb{R} \setminus \{0\}$. We have:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

Thus, we can define the extension by continuity of f at point $x_0 = 0$ as:

$$\tilde{f}(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

The function $\tilde{f}$ is continuous on $\mathbb{R}$.

6.6 Continuous Functions on Intervals

Definition 6.6.1 (Continuous Function on an Interval).
A function $f : I \to F$ where I is an open, non-empty interval, is said to be continuous on the interval I if it is continuous at every point $x_0 \in I$ i.e. for $f : [a, b] \to F$ to be continuous on $[a, b]$, it must be continuous on (a, b), continuous on the right at point a and continuous on the left at point b.

Example. Let's consider the function:

$$f(x) = \begin{cases} \frac{1}{2x+3}, & x \neq -\frac{3}{2} \\ 0, & x = -\frac{3}{2} \end{cases}$$

The function f is continuous on the interval $\left(-\infty, -\frac{3}{2}\right)$ and $\left(-\frac{3}{2}, +\infty\right)$ but not continuous at point $x = -\frac{3}{2}$ since the limit $\lim_{x \to -\frac{3}{2}} f(x)$ does not exist.

Theorem 6.6.1. Let $a < b \in \mathbb{R}$ and $f : [a, b] \to F$ be a continuous function on the bounded and closed interval $[a, b]$. Then, f reaches its minimum and maximum on $[a, b]$, i.e., there exist $c, d \in [a, b]$ such that:

$$f(c) = \min_{x \in [a,b]} f(x) \quad \text{and} \quad f(d) = \max_{x \in [a,b]} f(x)$$

Theorem 6.6.2 (Indeterminate Value Theorem)**.** Let $a < b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be a continuous function on the bounded and closed interval $[a, b]$. Then, for every L in $[f(a), f(b)]$, there exists at least one $c \in [a, b]$ such that:

$$f(c) = L$$

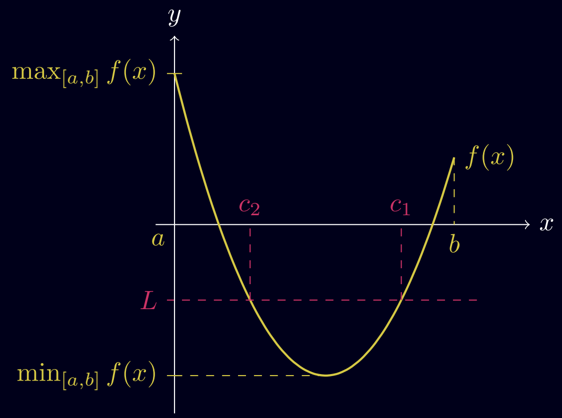
i.e.:

$$f([a, b]) = [\min_{x \in [a,b]} f(x), \max_{x \in [a,b]} f(x)]$$

Remark the following:

- If $f(a) \cdot f(b) < 0$, then there exists at least one $c \in [a, b]$ such that $f(c) = 0$.
- Given an open interval I and a strictly monotonic, continuous function $f : I \to \mathbb{R}$, the image $f(I)$ is an open interval.
- For all injective and continuous function on an interval I, the function is strictly monotonic.
- For all bijective and continuous function on an interval I, the inverse function is also continuous and strictly monotonic on the interval $f(I)$.

Example. Let's consider the following graph:



We have $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$. According to the Indeterminate Value Theorem, there exist at least a point (in this case two) $c_1, c_2 \in [a, b]$ such that $f(c_1) = f(c_2) = L$ for any $L \in [\min_{x\in[a,b]} f(x), \max_{x\in[a,b]} f(x)]$.

Example. Let's show that the equation $\sin x + \frac{1}{x-4} = 0$ has at least two solution in the interval $[0, \pi]$. We first notice that the function is continuous on the interval since it is the sum of two continuous functions on $[0, \pi]$ and the interval is bounded and closed, thus we can use the Intermediate Value Theorem. We have:

$$f(0) = \sin 0 + \frac{1}{0-4} = -\frac{1}{4} < 0$$

and

$$f(\pi) = \sin \pi + \frac{1}{\pi - 4} = 0 + \frac{1}{\pi - 4} < 0$$

Since both values are negative, we need to find a point where the function is positive. Let's try $x = \frac{\pi}{2}$:

$$f\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} + \frac{1}{\frac{\pi}{2} - 4} = 1 + \frac{1}{\frac{\pi}{2} - 4} > 0$$

Thus, by the Intermediate Value Theorem, there exists at least one $c_1 \in (0, \frac{\pi}{2})$ such that $f(c_1) = 0$ and at least one $c_2 \in (\frac{\pi}{2}, \pi)$ such that $f(c_2) = 0$. Therefore, the equation $\sin x + \frac{1}{x-4} = 0$ has at least two solutions in the interval $[0, \pi]$.

6.7 Exercices

This section gathers a selection of exercises related to Chapter 6, taken from weekly assignments, past exams, textbooks, and other sources. The origin of each exercise will be indicated at its beginning.

Exercise 6.7.1 (*Quizz of Lecture 13*).
Let $f(x) = 2\sin(1 - x^2)$ on the biggest interval where it is bijective and containing $x = 1$. Then the set of the definition of f^{-1} and its image is:

- $f^{-1} : [-2\sin(1), 2\sin(1)] \to \left[\sqrt{1-\frac{\pi}{2}}, \sqrt{1+\frac{\pi}{2}}\right]$
- $f^{-1} : [0, 2\sin(1)] \to \left[0, \sqrt{1+\pi}\right]$
- $f^{-1} : [-2, 2\sin(1)] \to \left[0, \sqrt{1+\frac{\pi}{2}}\right]$
- $f^{-1} : [-2, 2\sin(1)] \to \left[-\sqrt{1+\frac{\pi}{2}}, \sqrt{1+\frac{\pi}{2}}\right]$
- $f^{-1} : [-2\sin(1), 0] \to \left[0, \sqrt{1+\pi}\right]$

Answer:
The correct answer is the 3rd proposition because we have:

$$f(x) = 2\sin(1-x^2) \quad\implies\quad -\frac{\pi}{2} + 2k\pi \leq 1 - x^2 \leq \frac{\pi}{2} + 2k\pi$$

for $k = 0$ since $x = 1$ is in the interval. Thus:

$$1 - \frac{\pi}{2} \leq x^2 \leq 1 + \frac{\pi}{2} \quad\iff\quad 0 \leq x^2 \leq 1 + \frac{\pi}{2}$$

since $1 - \frac{\pi}{2} < 0$ and $x^2 \geq 0$. Therefore:

$$0 \leq x \leq \sqrt{1 + \frac{\pi}{2}}$$

Thus the domain of $f(x)$ is $\left[0, \sqrt{1+\frac{\pi}{2}}\right] = E$ for it to be bijective and containing $x = 1$. Furthermore, we see that f is decreasing on E thus:

$$\inf_{x\in E} f(x) = f\left(\sqrt{1+\frac{\pi}{2}}\right) = -2 \quad \text{and} \quad \sup_{x\in E} f(x) = f(0) = 2\sin(1)$$

Therefore, the image of f is $[-2, 2\sin(1)]$.

Exercise 6.7.2 (*Quizz of Lecture 14*).
Let's compute the limit:

$$\lim_{x\to 0} \frac{\sqrt{1+x^2}}{\sqrt{1+\left(\sin\frac{1}{x}\right)^2} + \sqrt{1-\left(\sin\frac{1}{x}\right)^2}}$$

Answer:
Since $x^2 \geq 0$, we have:

$$1 \leq \sqrt{1+x^2} \leq \underbrace{1 + \frac{1}{2}x^2}_{\to 1\ (x\to 0)}$$

Thus by the Sandwich Theorem, we conclude that $\lim_{x\to 0}\sqrt{1+x^2} = 1$. Let's take two sequences $(a_k) = \frac{1}{\pi k}$ and $(b_k) = \frac{1}{\frac{\pi}{2}+2\pi k}$ such that $\lim_{k\to\infty} a_k = b_k = 0$. We have:

$$\lim_{k\to\infty}\left(\sqrt{1+\left(\sin\frac{1}{a_k}\right)^2}+\sqrt{1-\left(\sin\frac{1}{a_k}\right)^2}\right) = \lim_{k\to\infty}\left(\sqrt{1+\underbrace{(\sin a_k)^2}_{=0}}+\sqrt{1-\underbrace{(\sin a_k)^2}_{=0}}\right)$$

$$\lim_{k\to\infty}(1+1) = 2$$

and:

$$\lim_{k\to\infty}\left(\sqrt{1+\left(\sin\frac{1}{b_k}\right)^2}+\sqrt{1-\left(\sin\frac{1}{b_k}\right)^2}\right) = \lim_{k\to\infty}\left(\sqrt{1+\underbrace{(\sin b_k)^2}_{=1}}+\sqrt{1-\underbrace{(\sin b_k)^2}_{=1}}\right)$$

$$\lim_{k\to\infty}(\sqrt{2}+0) = \sqrt{2}$$

Thus:

$$\lim_{k\to\infty} f(a_k) = \frac{1}{2} \quad \text{and} \quad \lim_{k\to\infty} f(b_k) = \frac{1}{\sqrt{2}}$$

Since the two limits are different, we conclude that the limit $\lim_{x\to 0} f(x)$ does not exist.

Chapter 7

Differential Calculus

7.1 Differentiability of Functions

Definition 7.1.1 (Derivative).
It is said that f is differentiable at a point $x_0 \in I$ if the following limit exists and is finite:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

The value $f'(x_0)$ is called the derivative of f at the point x_0.

Remark that if f is differentiable at x_0, then it can be written as:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + r(x)$$

where the remainder term $r(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$ satisfies:

$$\lim_{x \to x_0} \frac{r(x)}{x - x_0} = \lim_{x \to x_0} \left(\underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\to f'(x_0)} - f'(x_0) \right) = 0.$$

Definition 7.1.2 (Linear Approximation).
Let f be a function that is differentiable at a point $x_0 \in I$. The linear approximation of f at the point x_0 is defined by:

$$f(x) = f(x_0) + a(x - x_0) + r(x).$$

where $a \in \mathbb{R}$. The value a is called the slope of the linear approximation and is equal to the derivative of f at the point x_0, i.e. $a = f'(x_0)$.

Example. Let's compute the derivative of the function $f(x) = x^2$. Let $x_0 \in \mathbb{R}$. We have:

$$f'(x_0) = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0.$$

Therefore, the derivative of the function $f(x) = x^2$ is given by $f'(x) = 2x$ for all $x \in \mathbb{R}$.

Example. Let's compute the derivative of the function $f(x) = \cos x$. Let $x_0 \in \mathbb{R}$. We have:

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{\cos x - \cos x_0}{x - x_0} = \lim_{x \to x_0} \frac{\cos(x_0 + (x - x_0)) - \cos x_0}{x - x_0} \\ &= \lim_{x \to x_0} \frac{\cos x_0 \cos(x - x_0) - \sin x_0 \sin(x - x_0) - \cos x_0}{x - x_0} \\ &= \lim_{x \to x_0} \left[\frac{-\sin x_0 \cdot \sin(x - x_0)}{x - x_0} + \frac{\cos x_0 (\cos(x - x_0) - 1)}{x - x_0} \right] \\ &= \lim_{x \to x_0} \left[-\sin x_0 \cdot \underbrace{\frac{\sin(x - x_0)}{x - x_0}}_{\to 1} + \cos x_0 \cdot \underbrace{\frac{\left(-2\sin^2\left(\frac{x - x_0}{2}\right)\right)}{\left(\frac{x - x_0}{2}\right)^2}}_{\to -2} \cdot \underbrace{\frac{\left(\frac{x - x_0}{2}\right)^2}{x - x_0}}_{\to 0} \right] \\ &= -\sin x_0. \end{aligned}$$

Therefore, the derivative of the function $f(x) = \cos x$ is given by $f'(x) = -\sin x$ for all $x \in \mathbb{R}$.

Definition 7.1.3 (Derivative Function).
Let $f : E \to \mathbb{R}$ be a function differentiable on a set $D(f') \subset E$. The derivative function of f is defined as the function $f' : D(f') \to \mathbb{R}$ such that:

$$f(x) = f'(x) \quad \forall x \in D(f').$$

Theorem 7.1.1. A derivable function f at a point $x_0 \in I$ is continuous at x_0.

Proof. Let $f : I \to \mathbb{R}$ be a function derivable at a point $x_0 \in I$. We have:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} [f(x_0) + f'(x_0)(x - x_0) + r(x)] = f(x_0) + f'(x_0) \cdot 0 + \lim_{x \to x_0} r(x) = f(x_0).$$

Therefore, the function f is continuous at the point x_0. □

Remark that the converse is not true: a function can be continuous at a point but not derivable at that point.

7.1.1 Operations on Derivatives

Let $f, g : I \to \mathbb{R}$ be two functions derivable at a point $x_0 \in I$. The following operations can be performed on the derivatives of f and g at the point x_0:

- The function $\alpha f + \beta g$ is derivable at the point x_0 and:

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0).$$

- The function $f \cdot g$ is derivable at the point x_0 and:

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0).$$

- If $g(x_0) \neq 0$, then the function $\frac{f}{g}$ is derivable at the point x_0 and:

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{(g(x_0))^2}.$$

Proof. Let's prove each of these properties one by one.
1. Linearity We have:

$$\begin{aligned}
(\alpha f + \beta g)'(x_0) &= \lim_{x \to x_0} \frac{\alpha f(x) + \beta g(x) - \alpha f(x_0) - \beta g(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \left[\alpha \cdot \frac{f(x) - f(x_0)}{x - x_0} + \beta \cdot \frac{g(x) - g(x_0)}{x - x_0}\right] \\
&= \alpha f'(x_0) + \beta g'(x_0).
\end{aligned}$$

2. Product Rule We have:

$$\begin{aligned}
(f \cdot g)'(x_0) &= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \left[f(x) \cdot \frac{g(x) - g(x_0)}{x - x_0} + g(x_0) \cdot \frac{f(x) - f(x_0)}{x - x_0}\right] \\
&= f(x_0) \cdot g'(x_0) + g(x_0) \cdot f'(x_0).
\end{aligned}$$

3. Quotient Rule We have:

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x_0) &= \lim_{x \to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x - x_0} \\
&= \lim_{x \to x_0} \frac{f(x)g(x_0) - f(x_0)g(x)}{(x - x_0)g(x)g(x_0)} \\
&= \lim_{x \to x_0} \frac{g(x_0)[f(x) - f(x_0)] - f(x_0)[g(x) - g(x_0)]}{(x - x_0)g(x)g(x_0)} \\
&= \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.
\end{aligned}$$

□

7.1.2 Derivatives of Usual Functions

$f(x)$	$f'(x)$
c (constant function)	0
$x^n \ (n \in \mathbb{N})$	nx^{n-1}
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x = \frac{1}{(\cos x)^2}$

Proof. Let's prove some of these results.
2. $f(x) = x^n$**:** Let $x_0 \in \mathbb{R}$. By recurrence on n:
For $n = 0$, we have $f(x) = 1$ and $f'(x) = 0$.
Assume that the result is true for some $n \in \mathbb{N}$. We have:

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{x^{n+1} - x_0^{n+1}}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x^n + x^{n-1}x_0 + \ldots + x_0^n)}{x - x_0} \\ &= \lim_{x \to x_0} (x^n + x^{n-1}x_0 + \ldots + x_0^n) = (n+1)x_0^n. \end{aligned}$$

3. $f(x) = e^x$: Let $x_0 \in \mathbb{R}$. We have:

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{e^x - e^{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{e^{x_0}(e^{x - x_0} - 1)}{x - x_0} \\ &= e^{x_0} \cdot 1 = e^{x_0}. \end{aligned}$$

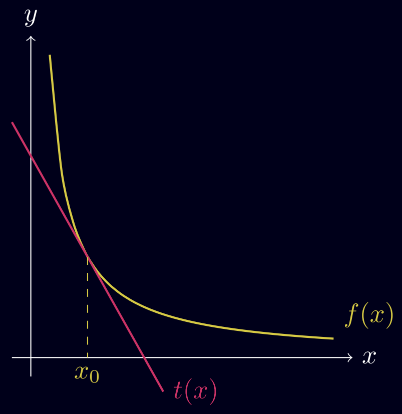
6. $f(x) = \tan x$**:** By the quotient rule, we have:

$$f'(x) = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{(\cos x)^2} = \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2}.$$

□

7.1.3 Geometric Interpretation of the Derivative

It can easily be seen from the definition of the derivative that the derivative at a point x_0 represents the slope of the tangent line $t(x)$ to the graph of the function f at the point $(x_0, f(x_0))$.



Definition 7.1.4 (Equation of the Tangent Line).
Let $f : I \to \mathbb{R}$ be a function differentiable at a point $x_0 \in I$. The equation of the tangent line $t(x)$ to the graph of f at the point $(x_0, f(x_0))$ is given by:

$$t(x) = f(x_0) + f'(x_0)(x - x_0)$$

7.1.4 Right and Left Derivatives

Definition 7.1.5 (Right and Left Derivatives).
Let $f : I \to \mathbb{R}$ be a function and $x_0 \in I$.
The right derivative of f at the point x_0 is defined as:

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

The left derivative of f at the point x_0 is defined as:

$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

Remark that if the right and left derivatives at a point x_0 exist and are equal, then the function is derivable at that point and the derivative is equal to the common value of the right and left derivatives, i.e.:

$$f'(x_0) = f'_+(x_0) = f'_-(x_0)$$

Example. Let's show that the function $f(x) = |x|$ is not derivable at the point $x_0 = 0$ (even if it is continuous on $\mathbb{R}$). We have:

$$f'(0) = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$$

We compute the left-hand limit and the right-hand limit:

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

Since the left-hand limit and the right-hand limit are not equal, the limit $\lim_{x \to 0} \frac{|x|}{x}$ does not exist. Therefore, the function $f(x) = |x|$ is not derivable at the point $x_0 = 0$.

7.1.5 Infinite Derivative

Definition 7.1.6 (Infinite Derivative)**.**
Let $f : I \to \mathbb{R}$ be a function and $x_0 \in I$. It is said that f has an infinite derivative at the point x_0 if:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \pm\infty$$

Remark that if a function has an infinite derivative at a point x_0, then the tangent line to the graph of the function at that point is vertical.

Example. Let $f(x) = \sqrt[3]{x}$. We are going to show that f has an infinite derivative at the point $x_0 = 0$. We have:

$$\lim_{x \to 0} \frac{\sqrt[3]{x} - \sqrt[3]{0}}{x - 0} = \lim_{x \to 0} \frac{\sqrt[3]{x}}{x} = \lim_{x \to 0} \frac{1}{\sqrt[3]{x^2}}$$

We compute the left-hand limit and the right-hand limit:

$$\lim_{x \to 0^-} \frac{1}{\sqrt[3]{x^2}} = +\infty$$

$$\lim_{x \to 0^+} \frac{1}{\sqrt[3]{x^2}} = +\infty$$

Since the left-hand limit and the right-hand limit are equal, the limit $\lim_{x \to 0} \frac{\sqrt[3]{x}}{x}$ is $+\infty$. Therefore, we can say that f has an infinite derivative at the point $x_0 = 0$. Therefore, the tangent line to the graph of the function at the point $(0, 0)$ is vertical.

7.1.6 Derivative of Composite Functions

Theorem 7.1.2 (Chain Rule)**.** Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions such that $f(I) \subset J$. If f is derivable at a point $x_0 \in I$ and g is derivable at the point $f(x_0) \in J$, then the composite function $g \circ f : I \to \mathbb{R}$ is derivable at the point x_0 and:

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

Proof. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions such that $f(I) \subset J$. We have:

$$\begin{aligned}(g \circ f)'(x_0) &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\ &= \lim_{x \to x_0} \left[\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}\right] \\ &= g'(f(x_0)) \cdot f'(x_0)\end{aligned}$$

□

Example. Let's compute the derivative of the function $f(x) = e^{2x^2 + \sin x}$. We have:

$$f'(x) = e^{2x^2 + \sin x} \cdot (4x + \cos x)$$

Example. Let's compute the derivative of the function $f(x) = \frac{\sin x}{e^x - e^{-x}}$. We have:

$$f'(x) = \frac{\cos x (e^x - e^{-x}) - \sin x (e^x + e^{-x})}{(e^x - e^{-x})^2}$$

7.1.7 Derivative of Inverse Functions

Theorem 7.1.3. Let $f : I \to F$ a bijective function that is continuous and differentiable at a point $x_0 \in I$ such that $f'(x_0) \neq 0$. Then, the inverse function $f^{-1} : F \to I$ is differentiable at the point $y_0 = f(x_0)$ and:

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

Proof. Let $f : I \to F$ a bijective function that is continuous and differentiable at a point $x_0 \in I$ such that $f'(x_0) \neq 0$. We have:

$$\begin{aligned}(f^{-1})'(y_0) &= \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} \\ &= \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}\end{aligned}$$

□

Remark that if $f : I \to F$ and $f^{-1} : F \to I$ two reciprocal and continuous function that are differentiable at every point of their respective domains, such that $f'(f^{-1}(x)) \neq 0$, then:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(y)} y = f^{-1}(x)\Big|_{y = f^{-1}(x)} \qquad \forall x \in F$$

Example. Let's compute the derivative of the function $f(x) = \arccos x$ ($[-1, 1] \to [0, \pi]$) by its inverse function $\cos x$ ($[0,\pi] \to [-1,1]$). We have:

$$f'(x) = \frac{1}{(\cos y)'|_{y=\arccos x}} = \frac{1}{-\sin(\arccos x)} = -\frac{1}{\sqrt{1-(\cos(\arccos x))^2}} = -\frac{1}{\sqrt{1-x^2}}$$

Remark that $\sin y = \pm\sqrt{1-\cos^2 y}$ and since $y \in [0,\pi]$, we have $\sin y \geq 0$ thus we choose the positive root.

Example. Let's compute the derivative of the function $f(x) = \ln x$ ($\mathbb{R}^+ \to \mathbb{R}$) by its inverse function e^x ($\mathbb{R} \to \mathbb{R}^+$). We have:

$$f'(x) = \frac{1}{(e^y)'|_{y=\ln x}} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Remark that $a^x = e^{x\ln a}$ for all $x \in \mathbb{R}$ and $a > 0, a \neq 1$. Thus, by the chain rule, we have:

$$(a^x)' = (e^{x\ln a})' = e^{x\ln a} \cdot \ln a = a^x \ln a$$

and by the inverse function theorem, we have:

$$(\log_a x)' = \frac{1}{(a^{\log_a x})'|_{x=\log_a x}} = \frac{1}{a^{\log_a x} \cdot \ln a} = \frac{1}{x\ln a}$$

Example. Let $r \in \mathbb{R}$ and $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ defined by $f(x) = x^r = e^{r\ln x}$. We have:

$$f'(x) = \left(e^{r\ln x}\right)' = e^{r\ln x} \cdot \frac{r}{x} = rx^{r-1}$$

Theorem 7.1.4. Let $f_1(x) > 0$ and $f_2(x)$ be two differentiable functions on an interval I. The function $f : I \to \mathbb{R}$ defined by $f(x) = f_1(x)^{f_2(x)}$ is differentiable on I and:

$$f'(x) = f_1(x)^{f_2(x)} \cdot (f_2(x) \cdot \ln(f_1(x)))'$$

Proof. Let $f_1(x) > 0$ and $f_2(x)$ be two differentiable functions on an interval I. We have:

$$f(x) = f_1(x)^{f_2(x)} = e^{f_2(x)\ln(f_1(x))}$$

Thus, by the chain rule, we get:

$$f'(x) = \left(e^{f_2(x)\ln(f_1(x))}\right)' = e^{f_2(x)\ln(f_1(x))} \cdot (f_2(x) \cdot \ln(f_1(x)))' = f_1(x)^{f_2(x)} \cdot (f_2(x) \cdot \ln(f_1(x)))'$$

□

More generally if $\ln(f(x))$ is defined, then:

$$(\ln f(x))' = \frac{f'(x)}{f(x)} \implies f'(x) = f(x) \cdot (\ln f(x))'$$

Example. Let's compute the derivative of the function $f(x) = x^{x^x}$. By the logarithm rule, we have:

$$f(x) = e^{x^x \ln x} = e^{e^{\ln x \cdot x} \ln x}$$

Thus, by the chain rule, we get:

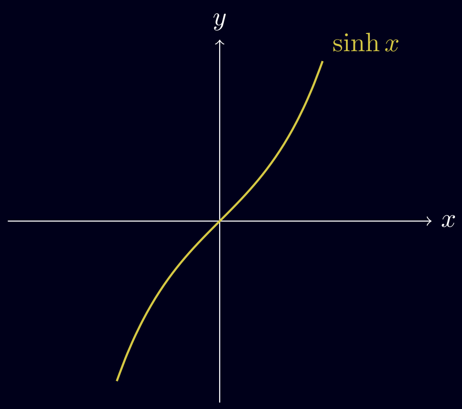
$$f'(x) = \left(e^{e^{\ln x \cdot x} \ln x}\right)' = x^{x^x} x^x \cdot \left[(\ln x)^2 + \ln x + \frac{1}{x}\right]$$

7.2 Hyperbolic Functions

Definition 7.2.1 (Sine and Cosine Hyperbolic Functions).
The hyperbolic sine function is defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \text{odd } \forall x \in \mathbb{R}$$

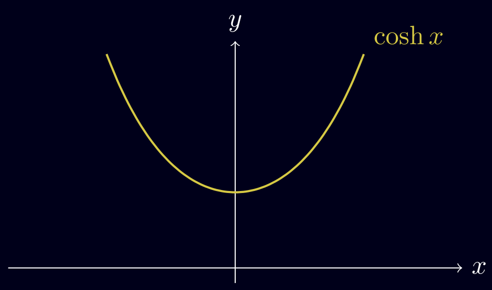
Graphically, the hyperbolic sine function can be represented as:



The hyperbolic cosine function is defined as follows:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \text{even } \forall x \in \mathbb{R}$$

Graphically, the hyperbolic cosine function can be represented as:



Remark that this definition is similar to the definition of the cosine and sine functions using

Euler's formula:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Some properties of hyperbolic functions are similar to those of trigonometric functions. For example, we have:

$$\cosh^2 x - \sinh^2 x = 1 \quad (\text{similar to } \cos^2 x + \sin^2 x = 1)$$

7.2.1 Derivatives of Hyperbolic Functions

Theorem 7.2.1. The derivatives of the hyperbolic sine and cosine functions are given by:

$$(\sinh x)' = \cosh x, \quad (\cosh x)' = \sinh x \quad \forall x \in \mathbb{R}$$

Proof. We have:

$$(\sinh x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$(\cosh x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{e^x - e^{-x}}{2} = \sinh x$$

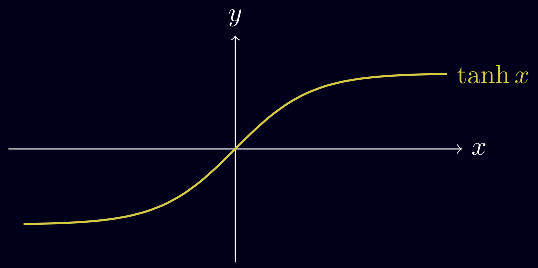
□

7.2.2 Tangent and Cotangent Hyperbolic Functions

Definition 7.2.2 (Tangent and Cotangent Hyperbolic Functions).
The hyperbolic tangent function is defined as follows:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \forall x \in \mathbb{R}$$

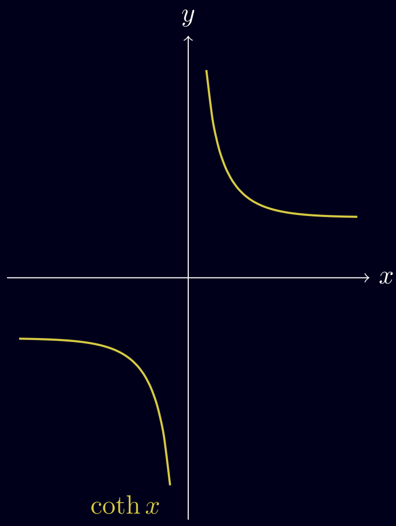
Graphically, the hyperbolic tangent function can be represented as:



The hyperbolic cotangent function is defined as follows:

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad \forall x \in \mathbb{R}^*$$

Graphically, the hyperbolic cotangent function can be represented as:



7.2.3 Arguments of Hyperbolic Functions

Definition 7.2.3 (Arguments of Hyperbolic Functions).
The inverse hyperbolic sine function is defined as follows:

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right) \quad \forall x \in \mathbb{R}$$

The inverse hyperbolic cosine function is defined as follows:

$$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right) \quad \forall x \geq 1$$

The inverse hyperbolic tangent function is defined as follows:

$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad \forall x \in (-1,1)$$

Theorem 7.2.2. The derivatives of the inverse hyperbolic functions are given by:

$$(\operatorname{arsinh} x)' = \frac{1}{\sqrt{x^2+1}}, \quad (\operatorname{arcosh} x)' = \frac{1}{\sqrt{x^2-1}}, \quad (\operatorname{artanh} x)' = \frac{1}{1-x^2}$$

Proof. We have:

$$(\operatorname{arsinh} x)' = \left(\ln\left(x+\sqrt{x^2+1}\right)\right)' = \frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}(x+\sqrt{x^2+1})} = \frac{1}{\sqrt{x^2+1}}$$

$$(\operatorname{arcosh} x)' = \left(\ln\left(x+\sqrt{x^2-1}\right)\right)' = \frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}(x+\sqrt{x^2-1})} = \frac{1}{\sqrt{x^2-1}}$$

$$(\operatorname{artanh} x)' = \left(\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right)' = \frac{1}{2}\cdot\frac{\frac{1}{1+x}+\frac{1}{1-x}}{\frac{1+x}{1-x}} = \frac{1}{2}\cdot\frac{(1-x)+(1+x)}{(1-x)(1+x)} = \frac{1}{1-x^2}$$

□

7.3 Derivatives of Higher Order

Definition 7.3.1 (Higher Order Derivatives).
Let $f : I \to \mathbb{R}$ be a function. If f is differentiable on I, then the derivative function $f' : I \to \mathbb{R}$ exists. If f' is also differentiable on I, then the second derivative function $f'' : I \to \mathbb{R}$ exists, and so on. The n-th derivative of f, denoted by $f^{(n)}$, is defined recursively as follows:

$$f^{(0)}(x) = f(x), \quad f^{(n)}(x) = (f^{(n-1)})'(x) \quad \forall n \in \mathbb{N}^*$$

Example. Let's compute the n-th derivative of the function $f(x) = x^n$ where $n \in \mathbb{N}^*$. We have:

$$f'(x) = nx^{n-1}, \quad f''(x) = n(n-1)x^{n-2}, \quad \ldots, \quad f^{(n)}(x) = n!$$

and for all $k > n$, we have:

$$f^{(k)}(x) = 0$$

Definition 7.3.2 (Class C^n Functions).
Let $n \in \mathbb{N} \cup \{\infty\}$. A function $f : I \to \mathbb{R}$ is said to be of class C^n on the interval I if it is n times differentiable on I and its n-th derivative $f^{(n)}$ is continuous on I.

Remark that the sine and cosine functions are of class C^∞ on $\mathbb{R}$ ($C^\infty(\mathbb{R})$) since they are differentiable any number of times and all their derivatives are continuous on $\mathbb{R}$.

7.4 Properties of Differentiable Functions

Theorem 7.4.1. If a function $f : E \to F$ is differentiable at a point $x_0 \in E$, such that f has a local extremum at x_0, then:

$$f'(x_0) = 0$$

Proof. Let $f : E \to F$ be a function differentiable at a point $x_0 \in E$, such that f has a local extremum at x_0. We have:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

If f has a local maximum at x_0, then for all x in a neighborhood of x_0, we have $f(x) \leq f(x_0)$. Thus, for $x > x_0$, we have:

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

and for $x < x_0$, we have:

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

Therefore, the right-hand limit is less than or equal to 0 and the left-hand limit is greater than or equal to 0. Since the limit exists, both limits must be equal, thus:

$$f'(x_0) = 0$$

The same reasoning applies if f has a local minimum at x_0. □

Remark that the converse is not true, i.e., if $f'(x_0) = 0$, it does not imply that f has a local extremum at x_0. For example, the function $f(x) = x^3$ has a derivative equal to 0 at the point $x_0 = 0$, but it does not have a local extremum at that point.

Definition 7.4.1 (Critical Point).
Let $f : E \to F$ be a function differentiable at a point $x_0 \in E$. The point x_0 is called a critical point of f if:

$$f'(x_0) = 0$$

Remark that the extrema of a function $f : [a, b] \to \mathbb{R}$ can only occur at:

- Critical points
- Points where the function is not differentiable
- Endpoints of the interval ($x = a$ or $x = b$)

Theorem 7.4.2 (Rolle's Theorem). Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b) such that:

$$f(a) = f(b)$$

Then, there exists at least one point $c \in (a, b)$ such that:

$$f'(c) = 0$$

Proof. Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$. By the Extreme Value Theorem, f attains its maximum and minimum values on the interval $[a, b]$. Let M be the maximum value of f on $[a, b]$ and let m be the minimum value of f on $[a, b]$:
If $M = m$, then f is constant on $[a, b]$, and thus $f'(x) = 0$ for all $x \in (a, b)$.
If $M \neq m$, then either the maximum or minimum value is attained at some point $c \in (a, b)$. Without loss of generality, assume that the maximum value is attained at some point $c \in (a, b)$. Since f is differentiable at c and has a local maximum at that point, by the previous theorem, we have:

$$f'(c) = 0$$

□

Example. Let's apply Rolle's theorem. Let $f : \mathbb{R} \to \mathbb{R}$ a function of class C^∞ such that:

$$f(0) = f(4) = 0 \quad \text{and} \quad f'(0) = f'(4) = 0$$

Let's show that the equation $f''(x) = 0$ has at least two solutions in the interval $(0, 4)$.
Since $f(0) = f(4) = 0$, by Rolle's theorem, there exists at least one point $c_1 \in (0, 4)$ such that:

$$f'(c_1) = 0$$

Since $f'(0) = f'(4) = 0$, by Rolle's theorem, there exists at least one point $c_2 \in (0, 4)$ such that:

$$f''(c_2) = 0$$

Now, we have two cases:

- If $c_1 < c_2$, then by Rolle's theorem, there exists at least one point $c_3 \in (c_1, c_2)$ such that:

$$f''(c_3) = 0$$

- If $c_1 > c_2$, then by Rolle's theorem, there exists at least one point $c_3 \in (c_2, c_1)$ such that:

$$f''(c_3) = 0$$

Therefore, in both cases, we have found at least two points c_2 and c_3 in the interval $(0, 4)$ such that:

$$f''(c_2) = 0 \quad \text{and} \quad f''(c_3) = 0$$