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## Analysis II

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# Chapter 1

## Differential Equations

### 1.1 Ordinary Differential Equations

#### 1.1.1 Definitions and Examples

**Definition 1.1.1 (Ordinary Differential Equation).**

An ordinary differential equation (ODE) is an equation that relates a function  $y(t)$  of a single variable  $t$  to its derivatives. The general form of an ODE can be expressed as:

$$F\left(t, y(t), y'(t), y''(t), \dots, y^{(n)}(t)\right) = 0$$

where  $F$  is a given function and  $n \in \mathbb{N}_+$  is the order of the ODE.

**Definition 1.1.2 (Solution of an ODE).**

A solution of the differential equation is a function  $y : I \rightarrow \mathbb{R}$  of class  $C^n$  defined on a open interval  $I \subset \mathbb{R}$  such that when substituted into the ODE, it satisfies the equation for all  $t \in I$ .

**Definition 1.1.3 (General Solution).**

The general solution of an ODE is a family of solutions that contains all possible solutions of the ODE. It typically includes arbitrary constants that can be adjusted to fit specific initial conditions or boundary conditions.

**Example.** Let  $y = y(x)$ ,  $x \in \mathbb{R}$  such that  $y' = 0$  for all  $x \in \mathbb{R}$ . Then  $y$  is a constant function, i.e.,  $y(x) = C$  for some constant  $C \in \mathbb{R}$ . This is the general solution of the ODE  $y' = 0$ . A particular solution can be obtained by specifying an initial condition, for example, if we set  $y(0) = 5$ , then the particular solution is  $y(x) = 5$  for all  $x \in \mathbb{R}$ .

**Example.** Let  $y'' = 3$  be the ODE. Integrating once, we get  $y' = 3x + C_1$  for some constant  $C_1 \in \mathbb{R}$ . Integrating again, we obtain  $y = \frac{3}{2}x^2 + C_1x + C_2$  for some constant  $C_2 \in \mathbb{R}$ . Thus,

the general solution of the ODE  $y'' = 3$  is given by:

$$y(x) = \frac{3}{2}x^2 + C_1x + C_2$$

where  $C_1$  and  $C_2$  are arbitrary constants. A particular solution can be found by specifying initial conditions, for example, if we set  $y(0) = 2$  and  $y'(0) = 1$ , we can determine the constants  $C_1$  and  $C_2$  to find the specific solution that satisfies these conditions.

**Example.** Let  $y + y' = 0$  be the ODE. We can rearrange this equation to get  $y' = -y$ . We easily verify that the function  $y(x) = Ce^{-x}$ , where  $C$  is an arbitrary constant, is a solution to this ODE on  $\mathbb{R}$ . Indeed, we have:

$$y'(x) = -Ce^{-x} = -y(x)$$

which satisfies the ODE. Thus, the general solution of the ODE  $y + y' = 0$  is given by:

$$y(x) = Ce^{-x}$$

where  $C$  is an arbitrary constant.

## 1.1.2 Terminology and Notation

Let  $E(x, y, \dots, y^{(n)}) = 0$  be a differential equation.

**Definition 1.1.4 (Order of a Differential Equation).**

The order of a differential equation is the highest order of the derivative that appears in the equation.

In the given equation, if  $y^{(n)}$  is the highest derivative present, then the order of the differential equation is  $n$ .

**Definition 1.1.5 (Linear Differential Equation).**

A differential equation is said to be linear if it can be expressed in the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x)$$

where  $a_n(x), a_{n-1}(x), \dots, a_0(x)$  and  $g(x)$  are given functions of  $x$  that are continuous.

**Definition 1.1.6 (Autonomous Differential Equation).**

A differential equation is called autonomous if it does not explicitly depend on the independent variable.

**Example.** Let's classify the following differential equations:

- $(2 \sin x)y + x^2y' = \cos x$  is a first-order linear non-autonomous ODE.
- $y^2 - y'' + y = 0$  is a second-order nonlinear autonomous ODE.

- $y''' + 3x^2y = e^x$  is a third-order linear non-autonomous ODE.
- $2xy = \sin y + y'$  is a first-order nonlinear non-autonomous ODE.

## 1.2 Separable Differential Equations

### Definition 1.2.1 (Separable Differential Equation).

A first-order ordinary differential equation is called separable if it can be expressed in the form:

$$f(y)y' = g(x)$$

where  $f : I \rightarrow \mathbb{R}$  and  $g : J \rightarrow \mathbb{R}$  are continuous functions defined on intervals  $I$  and  $J$  respectively.

Note that a solution  $y$  of a separable differential equation is a function  $y : J' \subset J \rightarrow I$  of class  $C^1$ .

**Example.** Let's come back to the previous equations  $y' = -y$ . This is a separable ODE since we can rewrite it as:

$$\underbrace{-\frac{1}{y}}_{f(y)} y' = \underbrace{1}_{g(x)} \implies \frac{1}{y} \frac{dy}{dx} = -1$$

We then can integrate both side to find the general solution:

$$\int \frac{1}{y} dy = \int -1 dx \implies \ln|y| + C_1 = -x + C_2 \implies |y| = e^{-x+C} = \underbrace{e^C}_{=C_3 > 0} e^{-x}$$

where  $C = C_2 - C_1$  is an arbitrary constant. Thus:

$$y = \underbrace{\pm C_3}_{\neq 0} e^{-x}$$

but we found in the previous example that  $y = 0$  is also a solution of the ODE. Therefore, the general solution of the ODE  $y' = -y$  is given by:

$$y(x) = Ce^{-x}$$

where  $C$  is an arbitrary constant that can take any real value, including zero.

## 1.3 Cauchy's Problem

### Definition 1.3.1 (Cauchy's Problem).

Cauchy's problem, also known as the initial value problem, is a type of problem in the theory of ordinary differential equations. It consists of finding a solution to a given ordinary differential equation that satisfies specified initial conditions. Formally, given an ODE of

the form:

$$F\left(t, y(t), y'(t), y''(t), \dots, y^{(n)}(t)\right) = 0$$

and initial conditions:

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \dots, \quad y^{(n-1)}(t_0) = y_{n-1}$$

where  $t_0$  is the initial time and  $y_0, y_1, \dots, y_{n-1}$  are given constants, the goal is to find a function  $y(t)$  and its interval of definition that satisfies both the ODE and the initial conditions.

**Definition 1.3.2 (Maximal Solution).**

A solution of an ODE is called maximal if there is no other bigger interval on which the function can be defined as a solution of the ODE.

**Example.** Let's solve  $y'' = 0$  with  $y(0) = 1$  and  $y'(0) = 2$ . Integrating once, we get  $y' = C_1$  for some constant  $C_1 \in \mathbb{R}$ . Integrating again, we obtain  $y = C_1x + C_2$  for some constant  $C_2 \in \mathbb{R}$ . Applying the initial conditions, we have:

$$y(0) = C_2 = 1$$

and

$$y'(0) = C_1 = 2$$

Thus, the particular solution that satisfies the initial conditions is given by:

$$y(x) = 2x + 1$$

for all  $x \in \mathbb{R}$ .

**Example.** Let's solve  $y'' = 0$  with  $y(0) = 3$  and  $y(4) = 1$ . Integrating once, we get  $y' = C_1$  for some constant  $C_1 \in \mathbb{R}$ . Integrating again, we obtain  $y = C_1x + C_2$  for some constant  $C_2 \in \mathbb{R}$ . Applying the initial conditions, we have:

$$y(0) = C_2 = 3$$

and

$$y(4) = 4C_1 + C_2 = 1 \quad \implies \quad C_1 = -\frac{1}{2}$$

Thus, the particular solution that satisfies the initial conditions is given by:

$$y(x) = -\frac{1}{2}x + 3$$

for all  $x \in \mathbb{R}$ .

**Theorem 1.3.1 (Existence and Uniqueness Theorem for Separable ODEs).** Let  $f : I \rightarrow \mathbb{R}$  be a continuous function such that  $f(y) \neq 0$  for all  $y \in I$  and  $g : J \rightarrow \mathbb{R}$  continuous. The for

any couple  $(x_0 \in J, b_0 \in I)$  the equation:

$$f(y)y' = g(x)$$

has a solution  $y : J' \subset J \rightarrow I$  verifying the initial condition  $y(x_0) = b_0$ . Moreover, if  $y_1$  and  $y_2$  are two solutions verifying the same initial condition, then  $y_1 = y_2$  on the intersection of their domains.

*Proof.* Let's prove the existence of a solution. Let:

$$F(y) = \int_{b_0}^y f(t) dt$$

Then  $F$  is differentiable and monotone since  $F'(y) = f(y) \neq 0$  on  $I$ . Thus,  $F(I)$  is a open interval and since:

$$b_0 \in I \implies F(b_0) = 0 \in F(I)$$

Thus  $F$  is a bijection of  $I$  in a neighbourhood of 0 and thus it admits an inverse function  $F^{-1}$  defined on a neighbourhood of 0. Let:

$$G(x) = \int_{x_0}^x g(t) dt$$

Again,  $G$  is differentiable on  $J$ ,  $G'(x) = g(x)$  and  $G(x_0) = 0$ . Let  $y(x) = F^{-1}(G(x))$  (defined on a neighbourhood of  $x_0$ ). We have:

$$F(y(x)) = G(x) \implies F'(y(x))y'(x) = G'(x) \implies f(y(x))y'(x) = g(x)$$

by the definition of  $F$  and  $G$ . Moreover, we have:

$$y(x_0) = F^{-1}(G(x_0)) = F^{-1}(0) = b_0$$

by the definition of  $F$  and  $G$ . Thus, we have found a solution  $y$  of the ODE that satisfies the initial condition.  $\square$

**Example.** Let's solve  $x^2y' = e^y$  with  $y(-2) = 0$ . We can rewrite the equation as:

$$\underbrace{e^{-y}}_{f(y)} y' = \underbrace{\frac{1}{x^2}}_{g(x)} \implies e^{-y} \frac{dy}{dx} = \frac{1}{x^2}$$

we have that  $f(y)$  is not equal to 0 for all  $y \in \mathbb{R}$  and  $g(x)$  is continuous on  $\mathbb{R} \setminus \{0\}$ . We can then integrate both sides to find the solution:

$$\int e^{-y} dy = \int \frac{1}{x^2} dx \implies -e^{-y} = -\frac{1}{x} + C \implies -y = \ln\left(\frac{1}{x} - C\right)$$

Thus:

$$y(x) = -\ln\left(\frac{1}{x} - C\right)$$

For this solution to exist, we need  $\frac{1}{x} - C > 0$ , which implies  $C < \frac{1}{x}$ . Let's find the intervals on which the solution is defined. We have three cases:

◦ If  $C > 0$ , then:

$$\frac{1}{x} > C \implies x > 0 \implies x < \frac{1}{C} \quad \text{and} \quad x > 0 \implies x \in \left(0, \frac{1}{C}\right)$$

◦ If  $C = 0$ , then:

$$\frac{1}{x} > 0 \implies x > 0 \implies x \in (0, \infty)$$

◦ If  $C < 0$ , then:

$$\frac{1}{x} \cdot \frac{1}{C} < 1 \implies \begin{cases} x > 0 \implies x > \frac{1}{C} \implies x \in (0, \infty) \\ x < 0 \implies x < \frac{1}{C} \implies x \in \left(-\infty, \frac{1}{C}\right) \end{cases}$$

Thus the general solution of the ODE  $x^2 y' = e^y$  is given by:

$$\begin{cases} y = -\ln\left(\frac{1}{x}\right) = \ln x, & x \in (0, \infty) \\ y = -\ln\left(\frac{1}{x} - C\right), & x \in \left(0, \frac{1}{C}\right), C > 0 \\ y = -\ln\left(\frac{1}{x} - C\right), & x \in (0, \infty), C < 0 \\ y = -\ln\left(\frac{1}{x} - C\right), & x \in \left(-\infty, \frac{1}{C}\right), C < 0 \end{cases}$$

Applying the initial condition  $y(-2) = 0$ , we have:

$$0 = -\ln\left(\frac{1}{-2} - C\right) \implies \frac{1}{-2} - C = 1 \implies C = -\frac{3}{2}$$

Thus, the particular solution that satisfies the initial condition is given by:

$$y(x) = -\ln\left(\frac{1}{x} + \frac{3}{2}\right)$$

which is defined on the interval  $\left(-\infty, -\frac{2}{3}\right)$ .

**Example.** Let's solve:

$$\frac{y'(x)}{y^2(x)} = 1$$

with  $y(0) = b_0 \in \mathbb{R}^*$ . We can rewrite the equation as:

$$\underbrace{\frac{1}{y^2}}_{f(y)} y' = \underbrace{1}_{g(x)} \implies \frac{1}{y^2} \frac{dy}{dx} = 1$$

we have that  $f(y)$  is not equal to 0 for all  $y \in \mathbb{R}^*$  and  $g(x)$  is continuous on  $\mathbb{R}$ . We can then integrate both sides to find the solution:

$$\int \frac{1}{y^2} dy = \int 1 dx \implies -\frac{1}{y} = x + C \implies y = -\frac{1}{x + C}$$

Applying the initial condition  $y(0) = b_0$ , we have:

$$b_0 = -\frac{1}{C} \implies C = -\frac{1}{b_0}$$

Which is the general solution defined on the intervals  $(-\infty, -C)$  and  $(-C, \infty)$ . The particular solution that satisfies the initial condition is given by:

$$y(x) = -\frac{1}{x - \frac{1}{b_0}} = \frac{1}{\frac{1}{b_0} - x}$$

To find the maximal interval on which this solution is defined, we need to find the values of  $x$  such that  $\frac{1}{b_0} - x \neq 0$ , which implies  $x \neq \frac{1}{b_0}$ . Thus, the maximal interval on which the solution is defined is given by:

$$\begin{cases} \left(-\infty, \frac{1}{b_0}\right), & b_0 > 0 \\ \left(\frac{1}{b_0}, \infty\right), & b_0 < 0 \end{cases}$$

Note that we chose the intervals to include the point  $x = 0$  in the maximal interval since it is the point at which the initial condition is defined.

## 1.4 First Order Linear Differential Equations

### Definition 1.4.1 (First Order Linear Differential Equation).

Let  $I \subset \mathbb{R}$  be an open interval. A first order linear differential equation is an equation of the form:

$$y' + p(x)y = f(x)$$

where  $p, f : I \rightarrow \mathbb{R}$  are continuous functions defined on  $I$ . A solution of this equation is a function  $y : I \rightarrow \mathbb{R}$  of class  $C^1$  that satisfies the equation.

### Definition 1.4.2 (Associated Homogeneous Equation).

The associated homogeneous equation of the first order linear differential equation  $y' + p(x)y = f(x)$  is the equation:

$$y' + p(x)y = 0$$

### 1.4.1 Principal of Superposition

**Theorem 1.4.1.** Let  $p, f_1, f_2 : I \rightarrow \mathbb{R}$  be continuous functions defined on an open interval  $I$ . If  $v_1$  is a solution of:

$$y' + p(x)y = f_1(x)$$

and  $v_2$  is a solution of:

$$y' + p(x)y = f_2(x)$$

Then for all  $C_1, C_2 \in \mathbb{R}$ , the function  $v = C_1v_1 + C_2v_2$  is a solution of:

$$y' + p(x)y = C_1f_1(x) + C_2f_2(x)$$

*Proof.* We have:

$$\begin{aligned} v' + p(x)v &= C_1v_1' + C_2v_2' + p(x)(C_1v_1 + C_2v_2) \\ &= C_1(v_1' + p(x)v_1) + C_2(v_2' + p(x)v_2) \\ &= C_1f_1(x) + C_2f_2(x) \end{aligned}$$

where we used the fact that  $v_1$  and  $v_2$  are solutions of their respective equations.  $\square$

### 1.4.2 Variation of Parameters

We want to find a particular solution of the equation:

$$y' + p(x)y = f(x)$$

of the form  $v(x) = C(x)e^{-P(x)}$ , where  $P(x)$  is a primitive of  $p(x)$  and  $C(x)$  is a function to be determined. We have:

$$\begin{aligned} v' + p(x)v &= C'(x)e^{-P(x)} + C(x) \left( -P'(x)e^{-P(x)} + p(x)e^{-P(x)} \right) \\ &= C'(x)e^{-P(x)} \\ &= f(x) \end{aligned}$$

which implies that  $C'(x) = f(x)e^{P(x)}$ . Integrating both sides, we get:

$$C(x) = \int f(x)e^{P(x)} dx$$

Thus, a particular solution of the equation  $y' + p(x)y = f(x)$  is given by:

$$v(x) = e^{-P(x)} \int f(x)e^{P(x)} dx$$

where  $P$  is a primitive of  $p$ .

**Theorem 1.4.2.** Let  $p, f : I \rightarrow \mathbb{R}$  be continuous functions. If  $v_0$  is a particular solution of  $y' + p(x)y = f(x)$ , then the general solution of this equation is given by:

$$v(x) = v_0(x) + Ce^{-P(x)}$$

where  $P$  is a primitive of  $p$  on  $I$  and  $C \in \mathbb{R}$ .

*Proof.* Let  $v_1$  be an arbitrary solution of the equation

$$y' + p(x)y = f(x)$$

Let's show that there exists  $C \in \mathbb{R}$  such that:

$$v_1(x) = v_0(x) + Ce^{-P(x)}$$

where  $v_0$  is a particular solution of the equation. We have by the principle of superposition that  $v_1 - v_0$  is a solution of the associated homogeneous equation  $y' + p(x)y = 0$ . Thus, there exists  $C \in \mathbb{R}$  such that:

$$v_1 - v_0 = Ce^{-P(x)} \implies v_1(x) = v_0(x) + Ce^{-P(x)}$$

Since  $v_1$  is an arbitrary solution of the equation, we have shown that any solution of the equation can be written in the form  $v(x) = v_0(x) + Ce^{-P(x)}$  for some  $C \in \mathbb{R}$  and thus by definition, that the general solution of the equation is given by  $v(x)$ .  $\square$

### 1.4.3 Solving First Order Linear Differential Equations

To solve a first order linear differential equation of the form  $y' + p(x)y = f(x)$ , we can follow these steps:

- Find the general solution of the associated homogeneous equation  $y' + p(x)y = 0$ .
- Find a particular solution of the original equation  $y' + p(x)y = f(x)$  using the method of variation of parameters.
- Combine the general solution of the homogeneous equation and the particular solution to get the general solution of the original equation.

**Example.** Let's find the general solution of the equation:

$$y' + \frac{1}{x}y = \frac{1}{x+1}$$

on the interval  $(0, \infty)$ . First, we find the general solution of the associated homogeneous equation  $y' + \frac{1}{x}y = 0$ . We can rewrite this equation as:

$$\frac{1}{y}y' = -\frac{1}{x}$$

Integrating both sides, we get:

$$\int \frac{1}{y} dy = \int -\frac{1}{x} dx \implies \ln |y| = -\ln |x| + K \implies |y| = e^{-\ln |x| + K} = \underbrace{e^K}_{=C > 0} \frac{1}{x}$$

Thus, the general solution of the homogeneous equation is given by:

$$y(x) = \frac{C}{x}$$

where  $C$  is an arbitrary constant. Next, we find a particular solution of the original equation using the method of variation of parameters. We have:

$$C(x) = \int f(x)e^{P(x)} dx = \int \frac{1}{x+1} e^{\ln x} dx = \int \frac{x}{x+1} dx$$

We can rewrite the integrand as:

$$\frac{x}{x+1} = \frac{(x+1) - 1}{x+1} = 1 - \frac{1}{x+1}$$

Thus, we have:

$$C(x) = \int \left(1 - \frac{1}{x+1}\right) dx = \int 1 dx - \int \frac{1}{x+1} dx = x - \ln|x+1| = x - \ln(x+1)$$

Thus, a particular solution of the original equation is given by:

$$y(x) = C(x)e^{-P(x)} = (x - \ln(x+1))e^{-\ln x} = 1 - \frac{\ln(x+1)}{x}$$

We can verify that this is indeed a solution of the original equation by substituting it back into the equation:

$$\begin{aligned} y' + \frac{1}{x}y &= \left(1 - \frac{\ln(x+1)}{x}\right)' + \frac{1}{x}\left(1 - \frac{\ln(x+1)}{x}\right) \\ &= \dots \\ &= \frac{1}{x+1} \end{aligned}$$

Thus, the general solution of the original equation is given by:

$$y(x) = 1 - \frac{\ln(x+1)}{x} + \frac{C}{x}$$

where  $C$  is an arbitrary constant.

## 1.5 Exercises

This section gathers a selection of exercises related to Chapter 1, taken from weekly assignments, past exams, textbooks, and other sources. The origin of each exercise will be indicated at its beginning.

### Exercise 1.5.1 (Course 2, Quiz).

The differential equation  $2yy' = 4x^3$  with the initial condition  $y(0) = 0$  has:

- No solution on  $\mathbb{R}$ .
- A unique solution on  $\mathbb{R}$ .
- Two distinct solutions on  $\mathbb{R}$ .
- Four distinct solutions on  $\mathbb{R}$ .

### Answer:

The correct answer is the fourth one, 4 distinct solutions on  $\mathbb{R}$ . Let's solve the equation.

We can rewrite it as:

$$\int 2y \, dy = \int 4x^3 \, dx \implies y^2 = x^4 + C \implies y = \pm\sqrt{x^4 + C}$$

with  $C \in \mathbb{R}$ . Now, we need to determine the value of  $C$  using the initial condition  $y(0) = 0$ . We have:

$$0 = y(0) = \pm\sqrt{0^4 + C} \implies C = 0$$

Therefore, the solutions to the initial value problem are:

$$y(x) = \pm\sqrt{x^4} = \pm x^2$$

There are four functions that satisfy the equation and the initial condition:

$$y_1(x) = x^2 \quad y_2(x) = -x^2 \quad y_3(x) = \begin{cases} -x^2 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases} \quad y_4(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ -x^2 & \text{if } x > 0 \end{cases}$$