École Polytechnique
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Analysis II

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Contents

Chapter 1

Differential Equations

1.1 Ordinary Differential Equations

1.1.1 Definitions and Examples

Definition 1.1.1 (Ordinary Differential Equation).
An ordinary differential equation (ODE) is an equation that relates a function $y(t)$ of a single variable t to its derivatives. The general form of an ODE can be expressed as:

$$F\left(t, y(t), y'(t), y''(t), \ldots, y^{(n)}(t)\right) = 0$$

where F is a given function and $n \in \mathbb{N}_+$ is the order of the ODE.

Definition 1.1.2 (Solution of an ODE).
A solution of the differential equation is a function $y : I \to \mathbb{R}$ of class C^n defined on a open interval $I \subset \mathbb{R}$ such that when substituted into the ODE, it satisfies the equation for all $t \in I$.

Definition 1.1.3 (General Solution).
The general solution of an ODE is a family of solutions that contains all possible solutions of the ODE. It typically includes arbitrary constants that can be adjusted to fit specific initial conditions or boundary conditions.

Example. Let $y = y(x)$, $x \in \mathbb{R}$ such that $y' = 0$ for all $x \in \mathbb{R}$. Then y is a constant function, i.e., $y(x) = C$ for some constant $C \in \mathbb{R}$. This is the general solution of the ODE $y' = 0$. A particular solution can be obtained by specifying an initial condition, for example, if we set $y(0) = 5$, then the particular solution is $y(x) = 5$ for all $x \in \mathbb{R}$.

Example. Let $y'' = 3$ be the ODE. Integrating once, we get $y' = 3x + C_1$ for some constant $C_1 \in \mathbb{R}$. Integrating again, we obtain $y = \frac{3}{2}x^2 + C_1 x + C_2$ for some constant $C_2 \in \mathbb{R}$. Thus, the general solution of the ODE $y'' = 3$ is given by:

$$y(x) = \frac{3}{2}x^2 + C_1 x + C_2$$

where C_1 and C_2 are arbitrary constants. A particular solution can be found by specifying initial conditions, for example, if we set $y(0) = 2$ and $y'(0) = 1$, we can determine the constants C_1 and C_2 to find the specific solution that satisfies these conditions.

Example. Let $y + y' = 0$ be the ODE. We can rearrange this equation to get $y' = -y$. We easily verify that the function $y(x) = Ce^{-x}$, where C is an arbitrary constant, is a solution to this ODE on $\mathbb{R}$. Indeed, we have:

$$y'(x) = -Ce^{-x} = -y(x)$$

which satisfies the ODE. Thus, the general solution of the ODE $y + y' = 0$ is given by:

$$y(x) = Ce^{-x}$$

where C is an arbitrary constant.

1.1.2 Terminology and Notation

Let $E(x, y, \dots, y^{(n)}) = 0$ be a differential equation.

Definition 1.1.4 (Order of a Differential Equation).
The order of a differential equation is the highest order of the derivative that appears in the equation.

In the given equation, if $y^{(n)}$ is the highest derivative present, then the order of the differential equation is n.

Definition 1.1.5 (Linear Differential Equation).
A differential equation is said to be linear if it can be expressed in the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x)$$

where $a_n(x), a_{n-1}(x), \dots, a_0(x)$ and $g(x)$ are given functions of x that are continuous.

Definition 1.1.6 (Autonomous Differential Equation).
A differential equation is called autonomous if it does not explicitly depend on the independent variable.

Example. Let's classify the following differential equations:

- $(2\sin x)y + x^2 y' = \cos x$ is a first-order linear non-autonomous ODE.
- $y^2 - y'' + y = 0$ is a second-order nonlinear autonomous ODE.

- $y''' + 3x^2 y = e^x$ is a third-order linear non-autonomous ODE.
- $2xy = \sin y + y'$ is a first-order nonlinear non-autonomous ODE.

1.2 Separable Differential Equations

Definition 1.2.1 (Separable Differential Equation).
A first-order ordinary differential equation is called separable if it can be expressed in the form:

$$f(y)y' = g(x)$$

where $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ are continuous functions defined on intervals I and J respectively.

Note that a solution y of a separable differential equation is a function $y : J' \subset J \to I$ of class C^1.

Example. Let's come back to the previous equations $y' = -y$. This is a separable ODE since we can rewrite it as:

$$\underbrace{-\frac{1}{y}}_{f(y)} y' = \underbrace{1}_{g(x)} \quad \Longrightarrow \quad \frac{1}{y}\frac{dy}{dx} = -1$$

We then can integrate both side to find the general solution:

$$\int \frac{1}{y} dy = \int -1 dx \quad \Longrightarrow \quad \ln|y| + C_1 = -x + C_2 \quad \Longrightarrow \quad |y| = e^{-x+C} = \underbrace{e^C}_{=C_3>0} e^{-x}$$

where $C = C_2 - C_1$ is an arbitrary constant. Thus:

$$y = \underbrace{\pm C_3}_{\neq 0} e^{-x}$$

but we found in the previous example that $y = 0$ is also a solution of the ODE. Therefore, the general solution of the ODE $y' = -y$ is given by:

$$y(x) = Ce^{-x}$$

where C is an arbitrary constant that can take any real value, including zero.

1.3 Cauchy's Problem

Definition 1.3.1 (Cauchy's Problem).
Cauchy's problem, also known as the initial value problem, is a type of problem in the theory of ordinary differential equations. It consists of finding a solution to a given ordinary differential equation that satisfies specified initial conditions. Formally, given an ODE of

the form:

$$F\left(t, y(t), y'(t), y''(t), \ldots, y^{(n)}(t)\right) = 0$$

and initial conditions:

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \ldots, \quad y^{(n-1)}(t_0) = y_{n-1}$$

where t_0 is the initial time and $y_0, y_1, \ldots, y_{n-1}$ are given constants, the goal is to find a function $y(t)$ and it's interval of definition that satisfies both the ODE and the initial conditions.

Definition 1.3.2 (Maximal Solution)**.**
A solution of an ODE is called maximal if there is no other bigger interval on which the function can be defined as a solution of the ODE.

Example. Let's solve $y'' = 0$ with $y(0) = 1$ and $y'(0) = 2$. Integrating once, we get $y' = C_1$ for some constant $C_1 \in \mathbb{R}$. Integrating again, we obtain $y = C_1x + C_2$ for some constant $C_2 \in \mathbb{R}$. Applying the initial conditions, we have:

$$y(0) = C_2 = 1$$

and

$$y'(0) = C_1 = 2$$

Thus, the particular solution that satisfies the initial conditions is given by:

$$y(x) = 2x + 1$$

for all $x \in \mathbb{R}$.

Example. Let's solve $y'' = 0$ with $y(0) = 3$ and $y(4) = 1$. Integrating once, we get $y' = C_1$ for some constant $C_1 \in \mathbb{R}$. Integrating again, we obtain $y = C_1x + C_2$ for some constant $C_2 \in \mathbb{R}$. Applying the initial conditions, we have:

$$y(0) = C_2 = 3$$

and

$$y(4) = 4C_1 + C_2 = 1 \quad \Longrightarrow \quad C_1 = -\frac{1}{2}$$

Thus, the particular solution that satisfies the initial conditions is given by:

$$y(x) = -\frac{1}{2}x + 3$$

for all $x \in \mathbb{R}$.

Theorem 1.3.1 (Existence and Uniqueness Theorem for Separable ODEs)**.** Let $f : I \to \mathbb{R}$ be a continuous function such that $f(y) \neq 0$ for all $y \in I$ and $g : J \to \mathbb{R}$ continuous. The for any couple $(x_0 \in J, b_0 \in I)$ the equation:

$$f(y)y' = g(x)$$

has a solution $y : J' \subset J \to I$ verifying the initial condition $y(x_0) = b_0$. Moreover, if y_1 and y_2 are two solutions verifying the same initial condition, then $y_1 = y_2$ on the intersection of their domains.

Proof. Let's prove the existence of a solution. Let:

$$F(y) = \int_{b_0}^{y} f(t)dt$$

Then F is differentiable and monotone since $F'(y) = f(y) \neq 0$ on I. Thus, $F(I)$ is a open interval and since:

$$b_0 \in I \quad \implies \quad F(b_0) = 0 \in F(I)$$

Thus F is a bijection of I in a neighbourhood of 0 and thus it admits an inverse function F^{-1} defined on a neighbourhood of 0. Let:

$$G(x) = \int_{x_0}^{x} g(t)dt$$

Again, G is differentiable on J, $G'(x) = g(x)$ and $G(x_0) = 0$. Let $y(x) = F^{-1}(G(x))$ (defined on a neighbourhood of x_0). We have:

$$F(y(x)) = G(x) \quad \implies \quad F'(y(x))y'(x) = G'(x) \quad \implies \quad f(y(x))y'(x) = g(x)$$

by the definition of F and G. Moreover, we have:

$$y(x_0) = F^{-1}(G(x_0)) = F^{-1}(0) = b_0$$

by the definition of F and G. Thus, we have found a solution y of the ODE that satisfies the initial condition. $\square$

Example. Let's solve $x^2y' = e^y$ with $y(-2) = 0$. We can rewrite the equation as:

$$\underbrace{e^{-y}}_{f(y)} y' = \underbrace{\frac{1}{x^2}}_{g(x)} \quad \implies \quad e^{-y}\frac{dy}{dx} = \frac{1}{x^2}$$

we have that $f(y)$ is not equal to 0 for all $y \in \mathbb{R}$ and $g(x)$ is continuous on $\mathbb{R} \setminus \{0\}$. We can then integrate both sides to find the solution:

$$\int e^{-y}dy = \int \frac{1}{x^2}dx \quad \implies \quad -e^{-y} = -\frac{1}{x} + C \quad \implies \quad -y = \ln\left(\frac{1}{x} - C\right)$$

Thus:

$$y(x) = -\ln\left(\frac{1}{x} - C\right)$$

For this solution to exist, we need $\frac{1}{x} - C > 0$, which implies $C < \frac{1}{x}$. Let's find the intervals on which the solution is defined. We have three cases:

- If $C > 0$, then:

$$\frac{1}{x} > C \quad \Longrightarrow \quad x > 0 \quad \Longrightarrow \quad x < \frac{1}{C} \quad \text{and} \quad x > 0 \quad \Longrightarrow \quad x \in \left(0, \frac{1}{C}\right)$$

- If $C = 0$, then:

$$\frac{1}{x} > 0 \quad \Longrightarrow \quad x > 0 \quad \Longrightarrow \quad x \in (0, \infty)$$

- If $C < 0$, then:

$$\frac{1}{x} \cdot \frac{1}{C} < 1 \quad \Longrightarrow \quad \begin{cases} x > 0 & \Longrightarrow \quad x > \frac{1}{C} \quad \Longrightarrow \quad x \in (0, \infty) \\ x < 0 & \Longrightarrow \quad x < \frac{1}{C} \quad \Longrightarrow \quad x \in \left(-\infty, \frac{1}{C}\right) \end{cases}$$

Thus the general solution of the ODE $x^2 y' = e^y$ is given by:

$$\begin{cases} y = -\ln(\frac{1}{x}) = \ln x, & x \in (0, \infty) \\ y = -\ln(\frac{1}{x} - C), & x \in \left(0, \frac{1}{C}\right), C > 0 \\ y = -\ln(\frac{1}{x} - C), & x \in (0, \infty), C < 0 \\ y = -\ln(\frac{1}{x} - C), & x \in \left(-\infty, \frac{1}{C}\right), C < 0 \end{cases}$$

Applying the initial condition $y(-2) = 0$, we have:

$$0 = -\ln\left(\frac{1}{-2} - C\right) \quad \Longrightarrow \quad \frac{1}{-2} - C = 1 \quad \Longrightarrow \quad C = -\frac{3}{2}$$

Thus, the particular solution that satisfies the initial condition is given by:

$$y(x) = -\ln\left(\frac{1}{x} + \frac{3}{2}\right)$$

which is defined on the interval $\left(-\infty, -\frac{2}{3}\right)$.

Example. Let's solve:

$$\frac{y'(x)}{y^2(x)} = 1$$

with $y(0) = b_0 \in \mathbb{R}^*$. We can rewrite the equation as:

$$\underbrace{\frac{1}{y^2}}_{f(y)} y' = \underbrace{1}_{g(x)} \quad \Longrightarrow \quad \frac{1}{y^2}\frac{dy}{dx} = 1$$

we have that $f(y)$ is not equal to 0 for all $y \in \mathbb{R}^*$ and $g(x)$ is continuous on $\mathbb{R}$. We can then integrate both sides to find the solution:

$$\int \frac{1}{y^2} dy = \int 1 dx \quad \Longrightarrow \quad -\frac{1}{y} = x + C \quad \Longrightarrow \quad y = -\frac{1}{x + C}$$

Applying the initial condition $y(0) = b_0$, we have:

$$b_0 = -\frac{1}{C} \implies C = -\frac{1}{b_0}$$

Which is the general solution defined on the intervals $(-\infty, -C)$ and $(-C, \infty)$. The particular solution that satisfies the initial condition is given by:

$$y(x) = -\frac{1}{x - \frac{1}{b_0}} = \frac{1}{\frac{1}{b_0} - x}$$

To find the maximal interval on which this solution is defined, we need to find the values of x such that $\frac{1}{b_0} - x \neq 0$, which implies $x \neq \frac{1}{b_0}$. Thus, the maximal interval on which the solution is defined is given by:

$$\begin{cases} \left(-\infty, \frac{1}{b_0}\right), & b_0 > 0 \\ \left(\frac{1}{b_0}, \infty\right), & b_0 < 0 \end{cases}$$

Note that we chose the intervals to include the point $x = 0$ in the maximal interval since it is the point at which the initial condition is defined.

1.4 First Order Linear Differential Equations

Definition 1.4.1 (First Order Linear Differential Equation).
Let $I \subset \mathbb{R}$ be an open interval. A first order linear differential equation is an equation of the form:

$$y' + p(x)y = f(x)$$

where $p, f : I \to \mathbb{R}$ are continuous functions defined on I. A solution of this equation is a function $y : I \to \mathbb{R}$ of class C^1 that satisfies the equation.

Definition 1.4.2 (Associated Homogeneous Equation).
The associated homogeneous equation of the first order linear differential equation $y' + p(x)y = f(x)$ is the equation:

$$y' + p(x)y = 0$$

1.4.1 Principal of Superposition

Theorem 1.4.1. Let $p, f_1, f_2 : I \to \mathbb{R}$ be continuous functions defined on an open interval I. If v_1 is a solution of:

$$y' + p(x)y = f_1(x)$$

and v_2 is a solution of:

$$y' + p(x)y = f_2(x)$$

Then for all $C_1, C_2 \in \mathbb{R}$, the function $v = C_1 v_1 + C_2 v_2$ is a solution of:

$$y' + p(x)y = C_1 f_1(x) + C_2 f_2(x)$$

Proof. We have:

$$\begin{aligned} v' + p(x)v &= C_1 v_1' + C_2 v_2' + p(x)(C_1 v_1 + C_2 v_2) \\ &= C_1(v_1' + p(x)v_1) + C_2(v_2' + p(x)v_2) \\ &= C_1 f_1(x) + C_2 f_2(x) \end{aligned}$$

where we used the fact that v_1 and v_2 are solutions of their respective equations. □

1.4.2 Variation of Parameters

We want to find a particular solution of the equation:

$$y' + p(x)y = f(x)$$

of the form $v(x) = C(x)e^{-P(x)}$, where $P(x)$ is a primitive of $p(x)$ and $C(x)$ is a function to be determined. We have:

$$\begin{aligned} v' + p(x)v &= C'(x)e^{-P(x)} + C(x)\left(-P'(x)e^{-P(x)} + p(x)e^{-P(x)}\right) \\ &= C'(x)e^{-P(x)} \\ &= f(x) \end{aligned}$$

which implies that $C'(x) = f(x)e^{P(x)}$. Integrating both sides, we get:

$$C(x) = \int f(x)e^{P(x)} dx$$

Thus, a particular solution of the equation $y' + p(x)y = f(x)$ is given by:

$$v(x) = e^{-P(x)} \int f(x)e^{P(x)} dx$$

where P is a primitive of p.

Theorem 1.4.2. Let $p, f : I \to \mathbb{R}$ be continuous functions. If v_0 is a particular solution of $y' + p(x)y = f(x)$, then the general solution of this equation is given by:

$$v(x) = v_0(x) + Ce^{-P(x)}$$

where P is a primitive of p on I and $C \in \mathbb{R}$.

Proof. Let v_1 be an arbitrary solution of the equation

$$y' + p(x)y = f(x)$$

Let's show that there exists $C \in \mathbb{R}$ such that:

$$v_1(x) = v_0(x) + Ce^{-P(x)}$$

where v_0 is a particular solution of the equation. We have by the principal of superposition that $v_1 - v_0$ is a solution of the associated homogeneous equation $y' + p(x)y = 0$. Thus, there exists $C \in \mathbb{R}$ such that:

$$v_1 - v_0 = Ce^{-P(x)} \quad \Longrightarrow \quad v_1(x) = v_0(x) + Ce^{-P(x)}$$

Since v_1 is an arbitrary solution of the equation, we have shown that any solution of the equation can be written in the form $v(x) = v_0(x) + Ce^{-P(x)}$ for some $C \in \mathbb{R}$ and thus by definition, that the general solution of the equation is given by $v(x)$. □

1.4.3 Solving First Order Linear Differential Equations

To solve a first order linear differential equation of the form $y' + p(x)y = f(x)$, we can follow these steps:

- Find the general solution of the associated homogeneous equation $y' + p(x)y = 0$.
- Find a particular solution of the original equation $y' + p(x)y = f(x)$ using the method of variation of parameters.
- Combine the general solution of the homogeneous equation and the particular solution to get the general solution of the original equation.

Example. Let's find the general solution of the equation:

$$y' + \frac{1}{x}y = \frac{1}{x+1}$$

on the interval $(0, \infty)$. First, we find the general solution of the associated homogeneous equation $y' + \frac{1}{x}y = 0$. We can rewrite this equation as:

$$\frac{1}{y}y' = -\frac{1}{x}$$

Integrating both sides, we get:

$$\int \frac{1}{y}dy = \int -\frac{1}{x}dx \quad \Longrightarrow \quad \ln|y| = -\ln|x| + K \quad \Longrightarrow \quad |y| = e^{-\ln|x|+K} = \underbrace{e^K}_{=C>0}\frac{1}{x}$$

Thus, the general solution of the homogeneous equation is given by:

$$y(x) = \frac{C}{x}$$

where C is an arbitrary constant. Next, we find a particular solution of the original equation using the method of variation of parameters. We have:

$$C(x) = \int f(x)e^{P(x)}dx = \int \frac{1}{x+1}e^{\ln x}dx = \int \frac{x}{x+1}dx$$

We can rewrite the integrand as:

$$\frac{x}{x+1} = \frac{(x+1)-1}{x+1} = 1 - \frac{1}{x+1}$$

Thus, we have:

$$C(x) = \int \left(1 - \frac{1}{x+1}\right) dx = \int 1dx - \int \frac{1}{x+1} dx = x - \ln|x+1| = x - \ln(x+1)$$

Thus, a particular solution of the original equation is given by:

$$y(x) = C(x)e^{-P(x)} = (x - \ln(x+1))e^{-\ln x} = 1 - \frac{\ln(x+1)}{x}$$

We can verify that this is indeed a solution of the original equation by substituting it back into the equation:

$$\begin{aligned} y' + \frac{1}{x}y &= \left(1 - \frac{\ln(x+1)}{x}\right)' + \frac{1}{x}\left(1 - \frac{\ln(x+1)}{x}\right) \\ &= \dots \\ &= \frac{1}{x+1} \end{aligned}$$

Thus, the general solution of the original equation is given by:

$$y(x) = 1 - \frac{\ln(x+1)}{x} + \frac{C}{x}$$

where C is an arbitrary constant.

1.5 Exercises

This section gathers a selection of exercises related to Chapter 1, taken from weekly assignments, past exams, textbooks, and other sources. The origin of each exercise will be indicated at its beginning.

Exercise 1.5.1 (*Course 2, Quizz*).
The differential equation $2yy' = 4x^3$ with the initial condition $y(0) = 0$ has:

- No solution on $\mathbb{R}$.
- A unique solution on $\mathbb{R}$.
- Two distinct solutions on $\mathbb{R}$.
- Four distinct solutions on $\mathbb{R}$.

Answer:
The correct answer is the fourth one, 4 distinct solutions on $\mathbb{R}$. Let's solve the equation.

We can rewrite it as:

$$\int 2y\,dy = \int 4x^3\,dx \quad\implies\quad y^2 = x^4 + C \quad\implies\quad y = \pm\sqrt{x^4 + C}$$

with $C \in \mathbb{R}$. Now, we need to determine the value of C using the initial condition $y(0) = 0$. We have:

$$0 = y(0) = \pm\sqrt{0^4 + C} \quad\implies\quad C = 0$$

Therefore, the solutions to the initial value problem are:

$$y(x) = \pm\sqrt{x^4} = \pm x^2$$

There are four functions that satisfy the equation and the initial condition:

$$y_1(x) = x^2 \quad y_2(x) = -x^2 \quad y_3(x) = \begin{cases} -x^2 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases} \quad y_4(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ -x^2 & \text{if } x > 0 \end{cases}$$